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On the existence and stability of two-dimensional Lorentzian tori without conjugate points

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Abstract

Infinitely many new examples of compact Lorentzian surfaces *without conjugate points* are given. Further, we study the existence and the stability of this property among Lorentzian metrics with a Killing field. We obtain a new obstruction and prove that the Clifton–Pohl torus and some of our examples are as stable as possible. This shows that in constrast with the Riemannian Hopf theorem, the absence of conjugate points in the Lorentzian setting is neither "special" nor rigid.

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1 Introduction

The absence of conjugate points on Riemannian tori has rigid effects on the metric structure. A result by E. Hopf in 1948 for a two-dimensional torus [8], and by Burago and Ivanov in 1994 in any dimension [6] states that any Riemannian torus with no conjugate points is necessarily flat. However, it appears that the Hopf theorem does not hold in the Lorentzian setting; in fact, Bavard and Mounoud proved in [1] that the so called Clifton-Pohl torus (see Eq. (3)) has no conjugate points. The Clifton-Pohl torus and its few natural deformations (see below) are the only known examples of Lorentzian metrics on the torus without conjugate points. Recall that any compact connected Lorentzian surface is homeomorphic to the torus or the Klein bottle. In this work, we give infinitely many new examples of geometrically non-equivalent Lorentzian tori and Klein bottles without conjugate points; moreover, we prove that some of them (including the Clifton–Pohl torus) admit a large space of deformations among metrics without conjugate points.

Given a non-flat torus T with a non-trivial Killing field K, the flow of K induces a free action of the group S^1 on T (see the proof of [2, Theorem 3.25]). The orbits of K are therefore periodic with the same period. Given the Clifton-Pohl metric, a first attempt to obtain deformations of such a metric without conjugate points can be achieved in three different ways: varying the period of the Killing field; acting by an homothety on the torus; or acting by a "twist" along a (closed) orbit of K. Although the variations above give non-isometric metrics on the torus without conjugate points, these examples are all locally "the same", having all the same universal cover (up to homothety). Less trivial deformations of the Clifton–Pohl torus without conjugate points are obtained by Mounoud in [9], as metrics projectively equivalent to the Clifton–Pohl torus; this gives a 2-dimensional family of Lorentzian tori without conjugate points, with non-isometric universal cover.

1.1 Lorentzian surfaces without conjugate points: sufficient conditions

The framework of this paper is that of compact Lorentzian surfaces admitting, like the Clifton–Pohl torus, a Killing vector field K. When K is timelike or spacelike, a result of Gutierrez, Palomo and Romero in [7] shows that if the surface has no conjugate points, it must be flat. In this paper, we suppose that K has a null orbit. When the metric is not flat, and K admits a null orbit, we call (T, K) a non-elementary torus.

In our first result, we show the existence of infinitely many new examples of compact Lorentzian surfaces without conjugate points. The examples we obtain are Lorentzian surfaces with a Killing field. The description of these metrics involves some non-trivial material from [2], which we shall introduce here in order to state properly the main results. Further details are left to Sect. 2 of this paper.

Let (T, K) be a non-flat torus, and let $\widetilde{T}$ be a universal cover of T, with $\widetilde{K}$ the lift of K to $\widetilde{T}$. Then K has no singularity (a Killing field of a compact Lorentzian surface does not vanish at any point, see [13, Lemma 4.1]). Locally, one of the two null foliations is defined

by a vector field L such that $\langle K, L \rangle = 1$, and the metric takes the form

$$2dxdy + f(x)dy^2,$$

where $L = \partial_x$, $K = \partial_y$, and f is a y-independent scalar function. The local coordinate x (defined up to translation and change of sign) is called a transverse coordinate. Hence $f(x)$ gives the norm of K in the transverse coordinate, and determines the metric locally. The globalization of this picture cannot be done in general, even in the universal cover (when none of the two null foliations is globally transverse to the Killing field). However, it appears in [2] that there exists a (unique up to translation and change of sign) global submersion $\mathbf{x} \in C^\infty(\widetilde{T}, \mathbf{R})$ with kernel the 1-dimensional vector space generated by $\widetilde{K}$, which coincides locally with a transverse coordinate. Hence the norm of $\widetilde{K}$ factorizes globally as follows

$$\langle \widetilde{K}, \widetilde{K} \rangle = f \circ \mathbf{x}, \tag{1}$$

where $f \in C^\infty(\mathbf{R}, \mathbf{R})$ is a non-constant periodic function. Replacing K by aK, $a \in \mathbf{R}^*$, and changing $\mathbf{x}$ induces a right action of the affine group of the line $\mathbf{R}^* \times \mathbf{R}$ on f which consists in replacing $f(\mathbf{x})$ by $a^2 f(a^{-1}\mathbf{x} + b)$, for $(a, b) \in \mathbf{R}^* \times \mathbf{R}$. We denote by $[f]$ the class of f modulo the action of $\mathbf{R}^* \times \mathbf{R}$.

Conversely, given a non-constant periodic function $f \in C^\infty(\mathbf{R}, \mathbf{R})$, we can associate to it a non-flat Lorentzian torus with a Killing field K, such that the norm of K factorizes as above. The way to define such a metric from f is described in Sect. 2.1.3. This metric however is not unique up to isometry. In [2, Theorem 5.29], the authors prove that if a non-elementary torus with a Killing field (T, K) has no conjugate points, then the foliation orthogonal to K is a (finite) union of Reeb components. When we restrict to metrics satisfying this additional property, given a non-constant periodic function $f \in C^\infty(\mathbf{R}, \mathbf{R})$ (modulo the action of $\mathbf{R}^* \times \mathbf{R}$), there exists a unique (up to isometry) non-flat metric on the plane which is the universal cover of a torus with a Killing field (T, K), such that $\langle \widetilde{K}, \widetilde{K} \rangle = f \circ \mathbf{x}$.

Now that we know this, we can state our first result

Theorem 1.1 *Let f be a periodic function that satisfies the following properties:*

- *i) f has simple zeros,*
- *ii) f' changes sign once between two sucessive zeros of f,*
- *iii) $f'.f''' \le 0$,*
- *iv) $\exists a \in \mathbf{R}, f(a + t) = f(a - t)$,*
- *v) f has two zeros in the smallest period of f.*

Then, a torus (T, K) such that the norm of K factorizes as in (1), and such that the foliation orthogonal to K is a union of Reeb components, has no conjugate points.

So whenever there is a function f which verifies the conditions of Theorem 1.1, we can associate to it a Lorentzian torus without conjugate points, and metrics we define from f_1 and f_2 are isometric if and only if $[f_1] = [f_2]$. Many explicit new (analytic) examples of Lorentzian tori without conjugate points, as well as Klein bottles with the same property, will be deduced from Theorem 1.1 in a very simple way.

The assumption (i) implies that the null orbits of $\widetilde{K}$ are isolated and that the norm of $\widetilde{K}$ changes sign when crossing such an orbit transversally (see [2, Lemma 2.25]). In particular, there exists only finitely many null orbits of K in T.

Remark 1.2 Each of these examples can be deformed to families of metrics without conjugate points, in the same way as previously done with the Clifton–Pohl torus (changing the period of the orbits of K, changing by homothety, twist or by projective deformation).

1.2 Lorentzian surfaces without conjugate points: necessary conditions

The structure of compact Lorentzian surfaces with a one parameter group of isometries is already studied in [2]. Although the property of being without conjugate points doesn't appear to be strong enough to expect a rigidity phenomenon in this subclass of Lorentzian surfaces (by rigidity phenomenon, we mean a strong consequence on the geometry of the surface, like Hopf's theorem in the Riemannian setting), [2, Theorem 5.29] gives obstructions for such tori to be without conjugate points. In particular, it follows from this theorem that a Lorentzian torus with a Killing field, without conjugate points, is either flat or non-homotopic to the flat metric. Assuming condition (i) in Theorem 1.1 (which is in some sense generic), we give a new obstruction for those tori to be without conjugate points. The leaves of the null foliations are geodesics. If such a leaf is closed and incomplete, then it is affinely isomorphic to a Hopf manifold of dimension one: $]0, \infty[/x \sim \lambda x$, λ playing the role of an affine length. In the case of a Clifton–Pohl torus (Eq. (3)), there are 4 (closed) null orbits of the Killing field; they are incomplete and they all have the same affine length. We extend this observation to the general case of a torus without conjugate points, admitting a Killing field whose null orbits are incomplete. We prove the following

Theorem 1.3 *Let (T, K) be a Lorentzian torus without conjugate points, with Killing vector field K. Assume that the null orbits of K are incomplete. Then they have the same affine length.*

As an immediate corollary of this theorem, it appears that these tori are limits of Lorentzian tori admitting a Killing field, and containing conjugate points (see Corollary 4.17 in this paper).

1.3 Strategy of the proof of theorem 1.1:

The Jacobi equation Let us recall that no null geodesic in a Lorentzian surface has conjugate points (see [10, page 291] for instance), so we restrict our attention to non-null geodesics. Let γ be a geodesic of an n-dimensional Lorentzian manifold M. A Jacobi field is a vector field along γ satisfying a differential equation called the Jacobi equation. There are many equivalent definitions of conjugate points; the one we will be using in this paper is the following: a pair of conjugate points on γ are points such that there exists a non-trivial Jacobi field along γ vanishing at these points. When γ is not lightlike, we can suppose that this vector field is orthogonal to γ; and when M is a surface, this reduces to the differential equation in one variable

$$u'' + \epsilon \kappa u = 0, \tag{2}$$

where κ is the sectional curvature along γ, and $\epsilon = \pm 1$ is the sign of $\langle \dot{\gamma}, \dot{\gamma} \rangle$.

If now M admits a Killing vector field K, one of the solutions of Eq. (2) is given by the normal component of K on γ. In the sequel, this solution will be denoted by β, the same notation used in [1].

Existence of maximal extensions Let (X, K) be a connected Lorentzian surface with a Killing field. A Lorentzian surface $(\hat{X}, \hat{K})$ is an extension of (X, K) if $\hat{X}$ is connected, and if there is an isometric embedding $\phi : X \to \hat{X}$ such that $\phi_*(K) = \hat{K}$. Such a surface is said to be maximal if it does not admit any non-trivial extension.

A Clifton–Pohl torus, denoted by T_{CP}, is the quotient of the manifold $\mathbf{R}^2 - \{0\}$ equipped with the metric

$$g_{CP} = \frac{2dxdy}{x^2 + y^2} \quad (x, y) \in \mathbf{R}^2 - \{0\}, \tag{3}$$

by some non trivial homothety.

The proof that T_{CP} has no conjugate points is done using a remarkable property: the universal cover of T_{CP} is a proper open subset of an extension $\hat{\Sigma}$, introduced in [1]. And this extension is geodesically complete. Let γ be a geodesic of $\widetilde{T}_{CP}$, and denote by $\widetilde{\gamma}$ its extension to $\hat{\Sigma}$. Some of the solutions of Eq. (2) defined over $\mathbf{R}$ vanish more than one time on $\widetilde{\gamma}$, but there are never two such zeros in the universal cover of the Clifton–Pohl torus. This is obtained through an explicit resolution of the Jacobi equations.

The possibility of extending the universal cover of a Lorentzian torus to a maximal Lorentzian surface is not specific to the Clifton–Pohl torus. Let (T, K) be a Lorentzian torus with a non-trivial Killing field K, and let $(\widetilde{T}, \widetilde{K})$ be its universal cover. There exists a maximal Lorentzian surface (E, K_E) homeomorphic to $\mathbf{R}^2$ such that $(\widetilde{T}, \widetilde{K})$ is isometrically embedded in E [2, Theorem 3.25]. This extension is unique when some further hypotheses are added on it; we do not recall that here. We prove that

Theorem 1.4 *The extension E associated to a Lorentzian torus (T, K) is geodesically complete.*

This makes these surfaces simple to deal with. When the torus is not flat, this extension always contains conjugate points (see [2, Proposition 5.28]). So one has to prove that there are never two such points in the domain in E corresponding to the universal cover of the torus, for the family given in Theorem 1.1 above. What makes this possible is the fact that this extension, precisely when (T, K) is a non-elementary torus, is a proper extension of the universal cover. Indeed, E is geodesically complete, but a non-flat torus with a Killing field K is complete if and only if K has no null orbit (this was observed by Sanchez in [13]). It appears in particular that the Lorentzian metrics without conjugate points obtained in Theorem 1.1 are geodesically incomplete. The existence of a complete and non-flat Lorentzian metric without conjugate points is still an open question.

In this paper, the existence of conjugate points is studied from the point of view of the oscillation theory of the Jacobi equation.

When condition iv) is added, we develop a rather simple point of view from which one can conclude to the existence or not of conjugate points in the torus. In this case, the only geodesics that could carry conjugate points are the (non-null) geodesics for which β vanishes (at least) twice in the extension. Given such a geodesic γ of the extension E (which is defined on $\mathbf{R}$ since E is complete), condition iv) is equivalent to the existence of some global isometry of E preserving K. This isometry induces some symmetries on β, and it implies in particular that the distance between two consecutive zeros of β is constant, equal to some $\tau > 0$. Furthermore, when the foliation orthogonal to K is a union of Reeb components (recall that this is a necessary condition to the absence of conjugate points), one proves easily that if γ intersects the domain in E corresponding to the universal cover of the torus, then it is contained there on an interval of type $I =]t_0, t_0 + \tau[$. Denote by V the vector space of the solutions of (2), and set

$$d := \inf_{\phi \in V \smallsetminus \{0\}} \{|t_0 - t_1|, \exists \phi \in V \smallsetminus \{0\}, \phi(t_0) = \phi(t_1) = 0\}.$$

The point is that there is no solution of (2) vanishing twice in I if and only if $d \geq \tau$. This amounts to saying that the absence of conjugate points is equivalent to the fact that β realizes

the minimum distance between two consecutive zeros of the solutions of the Jacobi equation. This is the point of view we use to prove Theorem 1.1 above. The distance between these zeros is studied by use of techniques from differential equation theory, provided in [3, 11] and [5]. The idea used is the fact that there exists a close connection between the oscillation problem of the Eq. (2) and the eigenvalue problem

$$u'' + \lambda \epsilon \kappa u = 0,$$

with suitable boundary conditions. This proof is an example of the interplay between geometry, analysis and the theory of ordinary differential equations in the study of conjugate points.

1.4 A large space of deformation

In the Riemannian setting, it is known from Hadamard's theorem that non-positive curvature implies that there are no conjugate points; therefore, it is easy to get open sets of Riemannian metrics without conjugate points in the C^2 topology. In the Lorentzian general setting, the stability question is a little harder; in fact, the restriction on the sign of the sectional curvatures is no help since the Jacobi equation involves also the type ϵ of the geodesic. If we drop the assumption concerning the additional symmetries on the universal cover of the torus, we show that the absence of conjugate points for a Lorentzian torus admitting a Killing field can be expressed in terms of the positivity of some numeric function defined on an open subset of the torus. The method used has the disadvantage of not providing a geometric ingredient to construct metrics without conjugate points, but it suggests that one can expect a stability result among the metrics admitting a Killing field. Denote by $\mathcal{L}_K(T)$ the space of smooth Lorentzian metrics on the 2-torus T admitting a non-trivial Killing field. We obtain

Theorem 1.5 *A metric in $\mathcal{L}_K(T)$ close enough to the Clifton-Pohl metric for the C^∞ topology, and satisfying the condition in Theorem 1.3 above, has no conjugate points.*

This theorem together with Theorem 1.3 above allow a very nice description of the neighborhood of T_{CP} in the closed (for the C^∞ topology) set of metrics without conjugate points admitting a Killing field: these are all neighbouring metrics whose null orbits have the same affine length.

Actually, we prove a more general stability by deformation result in the last section of this paper. This stability result ensures the existence of Lorentzian metrics without conjugate points and admitting a Killing field, without the symmetries added in Theorem 1.1.

The paper is organized as follows: in Sect. 2 we introduce the fundamental tools and notions from [2] dealing with the classification of compact Lorentzian surfaces with a Killing field, and prove the geodesic completeness of the maximal extensions associated to such surfaces. Sect. 3 is a study of the Jacobi equation regardless of geometry; we establish some lemmas about the distance between the zeros of the solutions of such an equation. These lemmas will be applied in Sect. 4 in the case of Lorentzian tori with a Killing field, in which we characterize Lorentzian tori without conjugate points in the way presented before. The obstruction in Theorem 1.3 will follow from this characterization. The last Section studies the stability character of the property of being without conjugate points, and prove that some of the examples obtained are as stable as possible in $\mathcal{L}_K(T)$.

2 Universal extensions of Lorentzian tori with a Killing vector field

2.1 Structure of Lorentzian tori with a Killing vector field

All the facts we will be compiling in this section, dealing with the geometry of Lorentzian tori admitting a Killing field, have been investigated in [2], in the more general setting of Lorentzian surfaces with a Killing field. For convenience, we set up in the first place some non-trivial vocabulary and notations, and then give the relevant results from [2] we will be using in this paper without proofs. Let (X, K) be a Lorentzian surface with a non-trivial and complete Killing field K.

2.1.1 Local structure and adapted coordinates

Definition 2.1 (ribbons, bands and dominoes) Let U be a subset of X saturated by K. Suppose K never vanishes on U; we say that (U, K) is

(1) a ribbon if U is open, simply connected and if one of the null foliations in U is everywhere transverse to K.
(2) a band (resp. a square) if U is homeomorphic to $[0, 1] \times \mathbf{R}$ (resp. $[0, 1]^2 \smallsetminus \{0, 1\}^2$), with $\langle K, K \rangle$ vanishing on the boundary and not vanishing in the interior of U.
(3) a domino if U is open, simply connected, and K has a unique null orbit in U.

The connected components of $X \smallsetminus \{\langle K, K \rangle = 0\}$ are interior of bands, and the interior of a band is a ribbon. A band is said to be spacelike (resp. timelike) if K is spacelike (resp. timelike) in this band.

Let (U, K) be a ribbon, and let $p \in U$. We can choose a null-geodesic γ passing through p, maximal in U and transverse to K, such that $\langle \gamma', K \rangle = 1$. The saturation of the geodesic by the flow of K gives a connected open set U_γ, which is equal to U when U is connected. This gives coordinates in U, called adapted coordinates, in which the metric reads

$$2dxdy + f(x)dy^2 \quad (x, y) \in I \times \mathbf{R}, \tag{4}$$

where $L = \partial_x$ is a null vector field parameterized by $\langle L, K \rangle = 1$ (and tangent to γ), and $K = \partial_y$. The coordinate denoted by x, which is well defined up to translation and change of sign, will be called the "transverse coordinate", or simply the x-coordinate. Thus, the norm of K in the ribbon is given by f in the x-coordinate; it vanishes on the null orbits of K contained in U.

The local chart in (4) cannot be globalized to X if and only if none of the two null foliations is everywhere transverse to K in X, which amounts to saying that K has two null orbits belonging to different null foliations. The latter fact is characterized in [2] by the presence of a Reeb component in the foliation defined by the Killing field or that of the orthogonal foliation.

Lemma 2.2 *[2, Lemma 2.8] Let (B, K) be a Lorentzian band. Denote by $\mathcal{L}$ a null foliation in B transverse to one of the two connected components of ∂B. If $\mathcal{L}$ is everywhere transverse to K in B, then the foliations defined by K and $K^\perp$ are both suspensions (admit global transversals). Otherwise, one of them is a Reeb component and the other is a suspension.*

Thus, we have the following definition, resulting from the lemma:

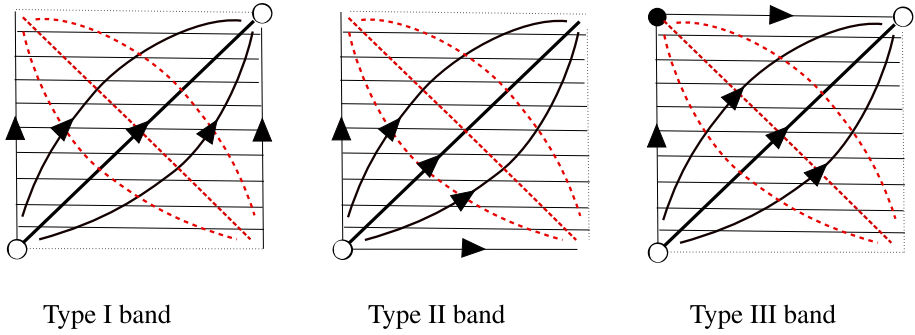**Definition 2.3** A Lorentzian band (B, K) is said to be:

Type I band   Type II band   Type III band

Fig. 1 A type I, type II and type III band

Fig. 2 The Clifton–Pohl torus has 4 bands of type II
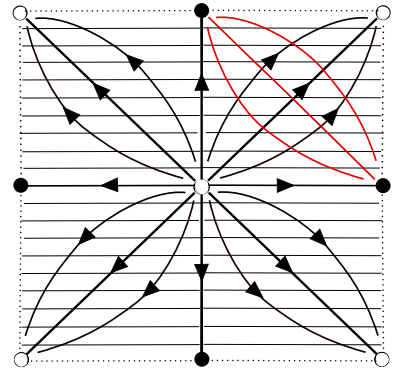
(1) of type I: if the foliations defined by K and $K^\perp$ are both suspensions.
(2) of type II: if the foliation of K is a suspension and that of $K^\perp$ is a Reeb component.
(3) of type III: if the foliation of K is a Reeb component and that of $K^\perp$ is a suspension.

The following figures represent the three types of bands; the foliation of K is represented by black continuous lines, the orthogonal foliation by red dotted lines.

Remark 2.4 1. If a ribbon contains a band, then it is necessarily a type I band (for the two null orbits of K on the boundary of a type II or a type III band belong to different null foliations).

2. A type I band contains one maximal ribbon. A type II or type III band is covered by two maximal ribbons whose intersection is the interior of the band.

3. Given a non-flat torus T with non-trivial Killing field K, the fact that the flow of K is a free action of S^1 ensures the existence of a closed curve everywhere transverse to K. It follows that T does not contain type III bands.

4. From 1) and 3) we have that a non-elementary torus (T, K) has only type II bands if and only if its maximal ribbons are dominos.

2.1.2 The existence of local reflections

The presence of a Killing field leads to the existence of local reflections. Let U be a connected component of $X \smallsetminus \{\langle K, K \rangle = 0\}$. In U, the geodesic flow orthogonal to K can be

parameterized by $\langle K^\perp, K \rangle = 0$, $\langle K^\perp, K^\perp \rangle = \pm 1$. The vector field thus obtained is invariant by K, i.e. $[K^\perp, K] = 0$. This gives coordinates $(u, v) \in I \times \mathbf{R}$ on U, in which the metric reads $\pm du^2 + h(u)dv^2$, where $K^\perp = \partial_u$, $K = \partial_v$. We observe that $(u, v) \mapsto (u, -v)$ is an isometry of U. This reflection sends K on $-K$ and permutes the two null foliations. It is called a "*generic reflection*".

So the reflection fixing a non-degenerate geodesic perpendicular to K is an isometry, but unlike the Riemannian case, if the norm of K vanishes somewhere, then the local reflection cannot be extended to the entire surface (see Fig. 2 above). In an adapted basis of the form (4), the leaves of the foliation orthogonal to K are defined by $y = G(x) + \beta$, where G is a primitive function of $-1/f$ and $\beta \in \mathbf{R}$ [2, Proposition 2.5]. And the generic reflections are given by $\phi(x, y) = (x, 2(G(x) + \beta) - y)$, $(x, y) \in I \times \mathbf{R}$.

Remark 2.5 Let U be the interior of a band. If U is contained in a type III band, then any generic reflection of U extends to the type III band, and sends a null leaf of K in the boundary to the other. If U is contained in a type I or type II band, the generic reflection cannot be extended.

An atlas for (X, K). If U denotes a ribbon in X and $p \in U$, we have seen above that the choice of a null-geodesic γ passing through p gives rise to adapted coordinates in which the metric writes

$$2dxdy + f(x)dy^2,$$

where $L = \partial_x$ is a null vector field parametrized by $\langle L, K \rangle = 1$, and $K = \partial_y$.

When $\langle K, K \rangle(p) \neq 0$, there exists another null-geodesic transverse to K and passing through p, giving rise to another formula for the metric on an open set U' of X. On the intersection $U \cap U'$, the norm of K doesn't vanish, so there exists a generic reflection that sends K on $-K$ and permutes the two null foliations. The transition map is given on $U \cap U'$ by composing $(x, y) \mapsto (-x, -y)$ with the generic reflection ϕ given by $\phi(x, y) = (x, 2G(x) - y)$, where G is a primitive function of $-1/f$.

This gives an atlas of X minus the zeros of K, such that on each local chart the metric reads $2dxdy + f(x)dy^2$, with $K = \partial_y$, and transition maps are like above. This atlas is called an **adapted atlas** for (X, K).

Example 2.6 **An atlas for the Clifton–Pohl torus.** Choosing the right parametrization of the null foliations, i.e. such that $\langle L, K \rangle = \langle L', K \rangle = 1$, we show that an atlas for the Clifton–Pohl torus is given by open sets $U_i = I_i \times \mathbf{R}$ ($i \in \mathbf{Z}/4\mathbf{Z}$), with $I_i =]\frac{i\pi}{2}, \frac{i\pi}{2} + \pi[$, each of them equipped with the metric $2dudv + sin(2u)dv^2$. A generic reflection on $U_i \cap U_{i+1}$ is given by $\phi(u, v) = (u, log|\frac{sin(2u)}{1-cos(2u)}| - v)$; it sends K to $-K$ and L to $-L'$. On two local charts U_i and U_{i+1} where the metric is given by $2dudv + sin(2u)dv^2$ and $2du'dv' + sin(-2u')dv'^2$ respectively, one gets a change of coordinates by composing $(u, v) \mapsto (-u, -v)$ with a generic reflection.

2.1.3 Classification of non-elementary tori

When there is no null orbit of K, the metric in the universal cover is determined by the norm of K like in the Riemannian case. If K has a null orbit, then the metric depends on additional combinatorial data (which determines the type of bands), described in [2, Proposition 5.2]. As mentioned in the introduction of this paper, there is a way to extract a non-constant periodic function $f \in C^\infty(\mathbf{R}, \mathbf{R})$ and some submersion $\mathbf{x} \in C^\infty(\widetilde{T}, \mathbf{R})$ such that $\langle \widetilde{K}, \widetilde{K} \rangle = f \circ \mathbf{x}$.

As we will need to understand how $\mathbf{x}$ and f depend on the metric in order to prove Lemma 2.11 and Lemma 2.12, we will give more details on their construction in this section.

Let (T, K) be a non-flat torus, and let $\widetilde{T}$ be a universal cover of T, with $\widetilde{K}$ the lift of K to the universal cover. As mentioned in the introduction of this paper, the norm of $\widetilde{K}$ can be written $\langle \widetilde{K}, \widetilde{K} \rangle = f \circ \mathbf{x}$, where $\mathbf{x} \in C^\infty(\widetilde{T}, \mathbf{R})$ is defined up to translation and change of sign, and $f \in C^\infty(\mathbf{R}, \mathbf{R})$ is a non-constant periodic function. To see this, consider a positively oriented hyperbolic basis (X, Y) in $\mathfrak{X}(\widetilde{T})$, i.e. $\langle X, X \rangle = \langle Y, Y \rangle = 0$ and $\langle X, Y \rangle = 1$. Define a volume form ν by setting $\nu(X, Y) = 1$ (ν does not depend on the choice of this basis). Define a 1-form $\omega := i_{\widetilde{K}} \nu$: this form is closed, hence exact since $\widetilde{T}$ is simply connected. So there exists a submersion $\mathbf{x} \in C^\infty(\widetilde{T}, \mathbf{R})$ defined (up to translation) by $\omega = d\mathbf{x}$, inducing a global diffeomorphism between the space of leaves of $\widetilde{K}$ and $\mathbf{R}$. Therefore, there exists a periodic function $f \in C^\infty(\mathbf{R}, \mathbf{R})$ such that the norm of $\widetilde{K}$ in the universal cover factorizes as before. The submersion $\mathbf{x}$ coincides, up to translation and change of sign, with the x-coordinate of any local chart.

Conversely, given a non-constant periodic function $f \in C^\infty(\mathbf{R}, \mathbf{R})$, we can associate to it a Lorentzian torus with a Killing field, such that the norm of K factorizes as above. This can be done in the following way: let (p_k) denote a sequence of zeros of f, taken in an increasing order, such that any interval $I_k =]p_k, p_{k+1}[$ contains at least one zero of f. Define ribbons $U_k = (I_k \times \mathbf{R}, (-1)^k 2 dx dy + f_k(x) dy^2)$, where $f_k = f_{|I_k}$. Then glue the open sets U_k and U_{k+1} by means of the local isometries $\psi_k(x, y) = (x, 2G(x) + y)$, where G is a primitive function of $-1/f$ on $I_k \cap I_{k+1}$. The surface we obtain is the universal cover of a torus, equipped with a Killing field given locally by $K = \partial_y$. The metric depends on the choice of $(p_k)_k$. Each time two ribbons U_k and U_{k+1} are glued by means of a generic reflection, the closure of their intersection is a type II band in the torus. It follows from Remark 2.4 that if the I_i's are chosen so as each of them contains only one zero of f, we get a surface all of whose bands are of type II. If we choose some I_i with an arbitrary number of zeros of f in it, the torus shall contain type I bands also.

A non-trivial fact from [2, Proposition 5.2] is that there is a bijection between isometry classes of universal cover of non-elementary tori (T, K), such that the foliation orthogonal to K is a (finite) union of Reeb components, and periodic non-constant functions $f \in C^\infty(\mathbf{R}, \mathbf{R})$ which vanish.

Given a non-flat torus (T, K), such that $f \in C^\infty(\mathbf{R}, \mathbf{R})$ is the function induced by the norm of K, the construction above gives an atlas for the universal cover of T. We say that we have a structure modeled on E^u_f, or E^u_f-structure, on $(\widetilde{T}, \widetilde{K})$ which, locally, is only determined by the function f. Such a structure exists on a connected and Lorentzian surface (X, K) each time the norm function of the Killing vector field factorizes as above (see [2, Proposition 3.19]).

Definition 2.7 (Minimal number of bands) We assume that the connected components of $\{f \neq 0\}$ don't accumulate. Define $\mathfrak{n}$ to be the number of zeros of f on the smallest period.

For the Clifton–Pohl torus we have $\mathfrak{n} = 2$; it is itself a two-sheeted covering of a torus containing 2 bands and having the same E^u_f structure as T_{CP}.

We won't consider the case in which these connected components accumulate since the locally finite hypothesis is a necessary condition for a torus admitting a Killing vector field to be without conjugate points (see [2, Theorem 5.29]).

2.1.4 Maximal extensions

The possibility of extending the universal cover of a Lorentzian torus into a Lorentzian surface all of whose null geodesics are complete (we say *L*-complete) will be used in an essential way in this paper; we call the Lorentzian surface so obtained the "universal extension" of the torus. We shall recall some essential properties of this surface. Let I be a non-empty open interval of $\mathbf{R}$ and let $f : I \to \mathbf{R}$ be a smooth function. Denote by $R_f = (R, K)$ the surface $(R = I \times \mathbf{R}, 2dxdy + f(x)dy^2)$, $(x, y) \in R$, with a Killing vector field $K = \partial_y$; it is called the "ribbon associated to f". R_{f^-} refers to the ribbon associated to f^-, defined for $-x \in I$ by $f^-(x) = f(-x)$.

Theorem 2.8 *([2, Proposition 3.3, Theorem 3.25]). Let (T, K) be a Lorentzian torus with a non-trivial Killing vector field K, $(\widetilde{T}, \widetilde{K})$ its universal cover. Let $f : \mathbf{R} \to \mathbf{R}$ denote the function induced by $\langle \widetilde{K}, \widetilde{K} \rangle$ in the way set before. There exists a maximal Lorentzian surface (E_f^u, K^u) homeomorphic to $\mathbf{R}^2$ such that $(\widetilde{T}, \widetilde{K})$ is isometrically embedded in E_f^u, and E_f^u is reflexive and L-complete. This extension is unique and every maximal ribbon contained in (E_f^u, K^u) is isometric to either R_f or R_{f^-}.*

Remark 2.9 Since f is periodic, when it vanishes, the connected components of the set $\{\langle K^u, K^u \rangle \neq 0\}$ in E_f^u are contained in squares. In a ribbon R_f of E_f^u, they are in one-to-one correspondance with the connected components of $\{f \neq 0\}$ in $\mathbf{R}$.

The Lorentzian surface E_f^u satisfies, in addition, the following remarkable reflexivity property:

Proposition 2.10 *([2, Lemma 3.10]). The generic reflections in any ribbon contained in E_f^u extend to global isometries of E_f^u.*

Let (T, K) be a Lorentzian torus with Killing field K, and let E_f^u be the extension of the universal cover, given in Theorem 2.8 above. This Lorentzian surface is obtained using two operations. Roughly speaking, they consist in:

1. gluing copies of R_f, the maximal ribbon defined above, along the bands, using generic reflections. This operation extends the null-geodesics interior to the bands of the ribbons into complete geodesics;

2. adding saddles: a saddle is obtained in [2, Proposition 2.29] as the extension of a domino of $\widetilde{T}$ (whose unique null orbit of $\widetilde{K}$ is incomplete) by a simply connected surface containing a unique zero of $\widetilde{K}$. This extends the null orbits of $\widetilde{K}$ which are geodesically incomplete into complete geodesics. With a good choice of the generic reflections in the first operation, one can make this extension compatible with the surface we get in 1 (see the proof of [2, Proposition 3.3] for details). Denote by U such a domino, and by $\widetilde{U}$ the extension. Write $U = I \times \mathbf{R}$, $2dxdy + f(x)dy^2$ in the local coordinates, with $f(0) = 0$ and $f'(0) = \lambda$, $\lambda \neq 0$ (Remark 4.15). The metric on $\widetilde{U}$ reads ([2, Proposition 2.29])

$$\frac{1}{\lambda} \left[v^2 h(uv)du^2 - 2\left(j(uv) + \frac{1}{j(uv)} \right) dudv + u^2 h(uv)dv^2 \right]; \quad uv \in I, v \in \mathbf{R}, \quad (5)$$

where $x = uv$; $j, h \in C^\infty(I, \mathbf{R})$, such that $j(x) = \int_0^1 f'(tx)dt$ and $h(x) = \int_0^1 l'(tx)dt$, with $l(x) = j(x) - \frac{1}{j(x)}$, and the Killing field corresponds to $K = \frac{2}{\lambda}(u\partial_u - v\partial_v)$.

We want to know how these objects, i.e. R_f and the saddles, depend on the metric. Actually, we will see that if the Killing field depends smoothly on the metric, then these objects do, too.

The space $\mathfrak{X}(T)$ of smooth vector fields on T, together with $\mathcal{L}_K(T)$, the space of smooth Lorentzian metrics on the 2-torus, admitting a Killing vector field, are equipped with the C^r topology.

Lemma 2.11 *Let $(g, K) \in \mathcal{L}_K(T) \times \mathfrak{X}(T)$ a non-flat metric on T, such that $\mathcal{L}_K g = 0$, and let (g_n, K_n) be a sequence in $\mathcal{L}_K(T) \times \mathfrak{X}(T)$ such that $\forall n$, $\mathcal{L}_{K_n} g_n = 0$ and $(g_n, K_n) \xrightarrow{C^r} (g, K)$. Then the sequence of ribbons R_{f_n}, where f_n denotes the function induced by the norm of K_n, converges to R_f, where f is induced by the norm of K, for the C^r topology.*

Proof For all n, one can choose a Lorentzian volume form ν_n that defines the transverse coordinate x_n by setting $i_{\widetilde{K}_n} \nu_n = dx_n$ and $\mathbf{x}_n(p) = 0$, such that the sequence ν_n converges to ν. By definition $\mathbf{x}_n$ converges C^r to $\mathbf{x}$ on every compact subset of $\widetilde{T}$. Now, take a curve c everywhere transverse to $\widetilde{K}$ and $\widetilde{K}_n$ for n large enough, cutting each leaf of $\widetilde{K}$ only one time. Define $I = \{c(t), t \in \mathbf{R}\}$ on which $\mathbf{x}$ and $\mathbf{x}_n$ are diffeomorphisms. Writing $f = g(\widetilde{K}, \widetilde{K}) \circ \mathbf{x}^{-1}$, we get the C^r convergence of f_n to f on every compact subset of $\mathbf{R}$, hence everywhere, since they are periodic. This proves the lemma.   □

Lemma 2.12 *Let (U, K), $U = I \times \mathbf{R}$, be a Lorentzian domino where the unique null orbit of K (represented by $x = 0$) is incomplete. Denote by g the metric on U and let (g_n, K_n) be a sequence of metrics on U such that $(g_n, K_n) \xrightarrow{C^r} (g, K)$, with $r \geq 2$. Then, there exists a neighborhood J of 0 such that the extension of $V := J \times \mathbf{R}$ for the metric g_n, denoted by $\widetilde{V}_n$, converges C^{r-2} to $\widetilde{V}$, the extension for the metric g.*

Proof Before starting the proof, let us state the following fact:

Fact 2.13 *let F_n be a sequence of functions defined on a compact manifold M with values in $\mathbf{R}$, which converges uniformly to a function F. Suppose that F admits a unique zero in M -denote it by p, and that F_n admits a unique zero p_n in M for all $n \in \mathbf{N}$. Then the sequence p_n converges to p in M.*

Now, let p be a point on the null orbit of K in U; the transverse coordinate for the metric g satisfies $x(p) = 0$. On a neighborhood of this orbit, there is a unique null orbit of K_n for all n but a finite number (the zero of f in U is simple by Remark 4.15), so let p_n be a point on it such that p_n converges to p (this is possible by Fact 2.13 above). Denote by $x_n \in C^\infty(U, I_n)$ the transverse coordinate of g_n such that $x_n(p_n) = 0$. Since I_n converges to I and f_n is C^r close to f, we can find a neighborhood J of 0 on which f and f_n are all defined for n big enough, with f_n having only one zero on J. The conclusion of the lemma follows then from Eq. (5) that gives the explicit expressions of the extensions.   □

Remark 2.14 We point out that these lemmas state the convergence of the ribbons and the saddles as abstract objects related to the metric, i.e. depending only on f. We didn't say anything about the local coordinates (x, y) and (u, v). Actually, we will see in Sect. 5.1 that these local charts depend smoothly on the metric.

2.2 Completeness of the universal extensions of non-elementary tori

We know from [1] that the extension associated to the Clifton–Pohl torus is complete. In this section, we prove that this is true for any extension E_f^u, with f periodic. Null-completeness being already obtained, what we have to show is that non-null geodesics are complete.

Theorem 2.15 *The extension E_f^u associated to a Lorentzian torus (X, K) is geodesically complete.*

The case in which f does not vanish follows from a result of Romero and Sanchez [12], who proved that a Lorentzian compact manifold (M, g) with a timelike Killing field is geodesically complete. In dimension 2, if g is a Lorentzian metric, then $-g$ is also a Lorentzian metric, so taking $-g$ if necessary, we can suppose, when f does not vanish, that $f < 0$, which gives the completeness.

Henceforth, we suppose that f vanishes. The length of a band (B, K) is the length of any maximal null geodesic α of B such that $\langle \alpha', K \rangle = 1$ (this is well defined for any two such geodesics are mapped to each other by an element of the flow of K composed possibly with a generic reflection of the band). Since f is periodic, the bands of E_f^u are contained in squares (Remark 2.9), hence the lengths of the bands are necessarily finite.

Let γ be a non-null geodesic in E_f^u. Denote by V the unit vector field tangent to γ, and N the vector field along γ orthogonal to V, such that the basis (V, N) is orthonormal and positively oriented. As mentioned in the introduction of this paper, we denote by β the normal component of K on γ; it is a solution of the Jacobi equation. Set

$$K^u = CV + \beta N.$$

Then $C = \epsilon \langle V, K^u \rangle$, where $\epsilon := \langle V, V \rangle$, is a constant called the Clairaut constant (see [1], p. 3), and $\beta = -\epsilon \langle K^u, N \rangle$ is a solution of the Jacobi equation. Notice that if $K^u(p)$, $p \in \gamma$, is not degenerate, $\beta(p) = 0$ if and only if γ is tangent to K^u at p. Finally, we have $\langle K^u, K^u \rangle = \epsilon(C^2 - \beta^2)$.

A maximal geodesic γ may have two different behaviors: either it leaves any maximal ribbon contained in E_f^u, or it remains in a maximal ribbon provided the geodesic parameter t goes close enough to the limit. In the second case, we shall consider two behaviors again: set $I = \{x(\gamma(t))\}$; as t ranges over the domain of γ, either I is bounded, in which case the geodesic remains in a band as t approaches the limit of the domain of γ, or I is unbounded. Observe that γ leaves a maximal ribbon if and only if it crosses a type II or a type III band, for a ribbon may have only type I bands (Remark 2.4).

Write $2dx\,dy + f(x)dy^2$ $(x \in I)$ for the metric in local coordinates. The equations $\langle \gamma'(t), \gamma'(t) \rangle = \epsilon$ and $\langle \gamma'(t), K^u \rangle = \epsilon C$ in the (x, y)-coordinates read:

$$2x'(t)y'(t) + f(x)y'(t)^2 = \epsilon \tag{6}$$

$$x'(t) + f(x)y'(t) = \epsilon C \tag{7}$$

We get

$$f(x)y'(t)^2 - 2\epsilon Cy'(t) + \epsilon = 0, \tag{8}$$

and

$$x'(t)^2 = C^2 - \epsilon f(x). \tag{9}$$

These equations shall be used in the proofs of the different lemmas that follow.

Lemma 2.16 *Let γ be a non-null geodesic such that β vanishes at most one time. If γ remains in a band for t large enough, it asymptotically approaches a leaf of K^u; if γ is not perpendicular to K^u, this leaf is either timelike or spacelike, depending on the type of γ.*

If a geodesic is somewhere orthogonal to K^u, then it is everywhere orthogonal to K^u, since the scalar product $\langle V, K^u \rangle$ is constant along a geodesic (Clairaut's constant). In consequence, the foliation orthogonal to K^u is a geodesic foliation.

Before we start the proof of the lemma, we need to make the following easy observation:
Observation: Let (R, K) be a ribbon with adapted coordinates $(x, y) \in I \times \mathbf{R}$, and let $p \in R$. Given $C \in \mathbf{R} \setminus \{0\}$, we have two cases:

1) $\langle K(p), K(p) \rangle = 0$: in this case, there is a unique $U \in T_p(R)$ such that $\langle K(p), U \rangle = C$ and $\langle U, U \rangle = 1$, and this vector is defined by $U = (C, \frac{1}{2C})$.
2) $\langle K(p), K(p) \rangle \neq 0$: there are two vectors $U_1, U_2 \in T_p(R)$ such that $\langle K(p), U \rangle = C$ and $\langle U, U \rangle = 1$. In the (x, y) coordinates, they are given by

$$U_1 = \left(\sqrt{C^2 - f(p)}, \frac{C - \sqrt{C^2 - f(p)}}{f(p)} \right), \tag{10}$$

$$U_2 = \left(-\sqrt{C^2 - f(p)}, \frac{C + \sqrt{C^2 - f(p)}}{f(p)} \right), \tag{11}$$

Only U_1 has positive first coordinate. And we have

C	$f(p)$	U_1^x	U_1^y
$+$	$+$	$+$	$+$
$+$	$-$	$+$	$+$
$-$	$+$	$+$	$-$
$-$	$-$	$+$	$+$

where (U_1^x, U_1^y) are the coordinates of U_1.

Proof Let $\gamma : [0, b[\to X$ be such a geodesic, with $\gamma([0, b[)$ contained in some band of X. Provided we go far enough out in γ, we may suppose that the geodesic is transverse to K^u in the band. If we see γ in the (x, y) coordinates, this amounts to saying that the derivative $x'(t)$ does not vanish on γ; the x-coordinate is therefore strictly monotone. Furthermore, $x(t)$ is bounded since γ is contained in a band (and the bands have finite length), hence converges to a constant x_0. Moreover, the y-coordinate is strictly monotone on γ ($y'(t)$ never vanishes for non-null geodesics). If y converges to y_0 on γ, it is easy to see that the geodesic can be extended beyond the point $p = (x_0, y_0)$. Indeed, following the proof of Lemma 8 p.130 [10], take a convex neighborhood $\mathcal{V}$ of p (an open set is said to be convex provided it is a normal neighborhood of each of its points). The geodesic γ is contained in $\mathcal{V}$ for $t \geq a$, for some $a > 0$; set $q = \gamma(a)$. In particular, there is a unique geodesic segment $\alpha : [0, 1] \to T$ joining p and q, that lies entirely in $\mathcal{V}$. This geodesic coincides with γ and extends it past p. It follows that the y-coordinate goes to infinity on γ while the geodesic approaches the leaf of K^u corresponding to x_0. Now, we want to prove that x' tends to 0. We start with the following fact:

Fact 2.17 *If the leaf of K^u corresponding to x_0 is not a null orbit of K^u, i.e. $f(x_0) \neq 0$, then x' tends to 0 on γ; in particular, $C^2 = \epsilon f(x_0)$.*

Indeed, assume that $f(x_0) \neq 0$; it appears from Eqs. (7) and (9) below that both $x'(t)$ and $y'(t)$ converge as x goes to x_0 on γ, with a finite limit. Now regard x as a function of y and write $\frac{dx}{dy} = \frac{x'(t)}{y'(t)}$; this derivative converges on γ; denote its limit by $l \in \bar{\mathbf{R}}$. Of course this limit cannot be infinite; indeed, x regarded as a function of y is strictly monotone and

tends to a constant while y goes to infinity; so if $\frac{dx}{dy}$ converges to l, this limit is necessarily zero. The conclusion that $x'(t)$ tends to zero is straightforward, and shows, using Eq. (9), that the leaf of K^u corresponding to x_0 has the same type as γ. It follows from this fact that if $f(x_0) \neq 0$ then γ asymptotically approaches the leaf of K^u corresponding to x_0.

Now, to prove that x' tends to 0 as t tends to b, we distinguish two different cases:

Case where $C = 0$: In this case, Eq. (9) below reads $x'(t)^2 = -\epsilon f(x)$. This yields $f(x_0) = 0$ using the previous fact, hence $x' \to 0$.

Case where $C \neq 0$: Without loss of generality we can assume that γ is spacelike. We will show that $f(x_0) \neq 0$, which will prove the lemma. We may suppose that $x' > 0$ on γ, by changing K^u to $-K^u$ in the local chart, if necessary. Suppose, contrary to our claim, that $f(x_0) = 0$, and denote by γ_∞ (resp. $\widetilde{\gamma}_\infty$) the corresponding null orbit of K^u (resp. the image of γ_∞ by a generic reflection of the band). In the type III band containing γ, γ_∞ and $\widetilde{\gamma}_\infty$, the space of the leaves of K^u is a simple branched line (thus a non-Hausdorff space), in which the branch points correspond to the null orbits of K^u: γ_∞ and $\widetilde{\gamma}_\infty$. The claim that x goes to x_0 on γ, with $f(x_0) = 0$, means that γ approaches one of the two branch points, i.e. either γ_∞ or $\widetilde{\gamma}_\infty$.

We distinguish two different behaviors of γ according to the sign of C. First, assume that $C > 0$; in this case, it follows from the observation preceding the proof that $x'(t), y'(t) > 0$ for all $t \in [0, b[$. We will show that γ (extends) and cuts γ_∞ transversally. Let q be a point on γ_∞, and denote by α the geodesic tangent to $V \in T_q(X)$, such that $\langle V, K^u(q) \rangle = C$ and $\langle V, V \rangle = 1$; it appears from the first part of the observation above that the two coordinates of α' at q are positive, hence remain positive all along the geodesic. Now take $\tau \in [0, b[$ such that for $t \geq \tau$, γ is transverse to K^u (the existence of τ is guaranteed by the assumption that β does not vanish for t close enough to the limit); denote by x_τ the coordinate of the orbit of K^u intersecting γ at $\gamma(\tau)$. From $C^2 = \beta^2 + f$, one gets $C^2 \geq \sup_{t \geq \tau} f(\gamma(t))$, hence $C^2 > \sup_{[x_\tau, x_0]} f(x)$. Using this together with Fact 2.17, we see that α, whose Clairaut's constant is also C, is defined and cuts all the leaves of K^u on the segment $[x_\tau, x_0]$ transversally. So by moving q on γ_∞ by the flow of K^u if necessary, one can suppose that γ and α intersect at a point p of the orbit of K^u having coordinate x_τ. At this point, we have $\langle \gamma'(p), K^u(p) \rangle = \langle \alpha'(p), K^u(p) \rangle = C$, and the coordinates of $\alpha'(p)$ are both positive. Now using the observation above, we see that in the interior of a band, only one of the two vectors satisfying $\langle U, K \rangle = C$ and $\langle U, U \rangle = 1$, has the additional property that the first coordinate is positive. This clearly forces $\alpha'(p) = \gamma'(p)$, hence γ extends beyond the band by cutting γ_∞ transversally. Of course this contradicts our assumption, and proves that actually, $f(x_0) > 0$.

Now, if $C < 0$, we prove that γ cuts $\widetilde{\gamma}_\infty$ transversally, leading to the same kind of contradiction as before. To deal with this case, define β to be the image of γ by a generic reflection ϕ of the band. Recall that $\phi(x, y) = (x, 2G(x) - y)$, where G is a primitive function of $-1/f$. Since ϕ sends K^u to $-K^u$, the Clairaut's constant of β is $C' := -C > 0$. Besides, $\beta' = \phi_* \gamma' = (x', \frac{-1}{f(x)}(2x' + f(x)y'))$ in the (x, y) coordinates. Write $\langle \gamma'(t), \gamma'(t) \rangle = y'(2x' + fy') = 1$. When $f < 0$ (resp. $f > 0$), we have $x' > 0$ and $y' > 0$ (resp. $x' > 0$ and $y' < 0$), hence $2x' + fy' > 0$ (resp. $2x' + fy' < 0$). It follows that in both cases, both coordinates of β' are positive. We are now in the same situation as before, so repeating the previous argument shows that β can be extended beyond the band by crossing γ_∞ transversally. Consequently, γ also extends beyond the band by cutting the image of γ_∞ by the generic reflection transversally. This leads again to a contradiction, and finishes the proof. $\qquad\square$

Proposition 2.18 *Let γ be a non-null geodesic such that β vanishes at most one time. If γ lies in a band after a certain while, it is complete.*

Proof According to the previous lemma, the geodesic asymptotically approaches a leaf of K^u.

Eq. (9) yields

$$t(x_0) - t(x) = \int_x^{x_0} \frac{dx}{\sqrt{C^2 - \epsilon f(x)}}.$$

Suppose $C = 0$; in this case, x_0 is not a simple zero of f, for if x_0 is a simple zero of f, the corresponding null orbit of K^u is extended in E_f^u by adding a saddle point (see [2] for details), and γ leaves the band through it. Thus the above integral goes to infinity. Now, if $C \neq 0$, γ approaches a non-null orbit of K^u so that $f(x)$ is bounded away from 0 as γ goes to this orbit; it follows from (8) that $y'(t)$ is bounded. Since y goes to infinity on γ, the latter is necessarily complete. □

Lemma 2.19 *Let γ be a non-null geodesic not orthogonal to K^u. Assume γ cuts a null orbit of K^u, denoted by $\mathfrak{l}$. Then*

 (i) *γ does not cross a type II band containing $\mathfrak{l}$;*
 (ii) *if γ crosses a type III band containing $\mathfrak{l}$, γ is tangent to K in that band;*
(iii) *if β does not vanish, γ lies in the maximal ribbon containing $\mathfrak{l}$.*

 Proof (i) By definition, the foliation orthogonal to K^u in a type II band is a Reeb component, so a geodesic that crosses a type II band is tangent somewhere to a leaf of the orthogonal foliation, hence coincides with it, since the leaves of this foliation are geodesics.
(ii) If γ crosses a type III band containing $\mathfrak{l}$, β vanishes in that band. Indeed, the foliation of K^u in a type III band is a Reeb component, so a geodesic that crosses a type III band is tangent to K^u somewhere. (iii) By (ii), γ doesn't cross a type III band containing $\mathfrak{l}$. It doesn't cross a type II band either by (i), so it lies in the maximal ribbon containing $\mathfrak{l}$. □

We now go back to the proof of Theorem 2.15:

Proof When f does not vanish, i.e. K^u is either timelike or spacelike, the torus is obviously complete; indeed, in this case, $|\inf\langle K^u, K^u\rangle| > 0$, and for any non-null geodesic $\gamma : [0, b[\to T$, Eqs. (7) and (9) imply that the closure of the image of γ' in $T(T)$ is compact. The same reasoning can actually be adapted to null geodesics, so that $E_f^u = \widetilde{T}$. More generally, if M is a compact Lorentzian manifold, with a timelike (or spacelike) Killing vector field, then M is complete, like in the Riemannian case. So now, assume that f changes sign.

1st **case: β vanishes at least twice on γ**
Under this assumption, γ is preserved by a translation of the parameter t of the geodesic; we call it periodic (see Lemma 4.8 for a proof); this gives a complete geodesic.

It appears from the proof of Lemma 4.8 that either γ remains in a band or it leaves any maximal ribbon of E_f^u. The first situation occurs when β vanishes twice in the interior of a band; in this case, γ is invariant by the action ϕ^τ of the flow of K^u, for a certain $\tau \in \mathbf{R}$.

2nd **case: β vanishes at most once on γ**
In this case, γ remains in a maximal ribbon after a certain while, by Lemma 2.19 above. If β vanishes, set $t = 0$ at this point; if it doesn't vanish, take an arbitrary point p on γ and set

$t = 0$ at this point. Let $I = \{x(\gamma(t)), t > 0\}$. Suppose that I is unbounded. Fix a band B_0 in the ribbon; in this band

$$t(x_1) - t(x_0) \geq \int_{x_0}^{x_1} \frac{1}{\sqrt{C^2 - m_0}} \, dx = a_0,$$

with $m_0 = \inf\limits_{B_0} \epsilon f \leq 0$. Now recall that f is periodic, so γ crosses infinitely many such bands (isometric to B_0), and since the time γ takes to cross a band depends only on C and f on this band, the conclusion follows easily.

We are left with the case in which I is bounded. In this case, γ remains in a band for t close enough to the limit; the completeness of γ follows from Proposition 2.18. $\square$

Remark 2.20 This result holds for any extension E_f^u, with f bounded.

Proof All we have to check is the completeness for a geodesic γ that remains in a maximal ribbon, with unbounded I. We have from the previous proposition

$$t(x_1) - t(x_0) = \int_{x_0}^{x_1} \frac{dx}{\sqrt{C^2 - \epsilon f(x)}}, \quad x_1 > x_0$$

where x_0 is the coordinate of an orbit of K^u cutting γ. Since f is bounded, we have

$$t(x_1) - t(x_0) > \int_{x_0}^{x_1} \frac{dx}{\sqrt{C^2 + M}},$$

where M is a majorant of f on **R**. Now, I is assumed to be unbounded; therefore, x_1 tends to infinity on γ, so that the above integral goes to infinity as $x_1 \to \infty$, which is the desired conclusion. $\square$

3 Study of the Jacobi equation

Given a Lorentzian surface with a Killing vector field K, the Jacobi equation along a non-null geodesic γ writes:

$$u'' + \epsilon \kappa u = 0, \tag{12}$$

where $\kappa(t)$ is the curvature along the geodesic $\gamma(t)$, $t \in I$. The restriction on the sign of the sectional curvature in the Riemannian setting leads to deep knowledge about the dynamics of the geodesic flow, through the behavior of the Jacobi fields. Unfortunately, this hypothesis makes no sense in the Lorentzian setting, since the quantity involved in the Jacobi equation in this case is $\epsilon \kappa$, where ϵ is the type of the geodesic. So in this Section, we shall investigate some properties of the solutions of the differential Eq. (12) under certain restrictive assumptions on the function $\kappa(t)$. The results will be applied in the case of a Lorentzian torus with a Killing vector field in the next section.

Lemma 3.1 *(see [1, Remark 1.3]). Given two independent solutions of the Jacobi equation, between two zeros of one solution, there is one and only one zero of the other.*

Now, denote by s and c the linearly independent solutions of (12) satisfying

$$\begin{cases} s(0) = 0 \\ s'(0) = 1 \end{cases} \quad \text{and} \quad \begin{cases} c(0) = 1 \\ c'(0) = 0 \end{cases}$$

It's easy to check that $(cs' - c's)(t) = (cs' - c's)(0) = 1$ for all $t \in I$. Assume $\kappa(t)$ is periodic, of period 2τ. Denote by A the endomorphism of the vector space V of the solutions of (12) (generated by s and c), and given by

$$A : V \longrightarrow V$$
$$u(t) \longmapsto u(t + 2\tau)$$

Eq. (12) has a periodic solution if and only if the endomorphism A has eigenvalue 1; in this case, the periodic solutions of the equation are exactly the eigenvectors associated with this eigenvalue.

The zeros of a non-trivial solution ϕ of (12) are simple, because $\phi(a) = \phi'(a) = 0$ for some a implies $\phi = 0$. Now, let $\zeta_1 < \zeta_2$ be two consecutive zeros of a solution of (12). By Lemma 3.1, there is a unique zero ζ of s in $[\zeta_1, \zeta_2[$. Denote by P the set of all such pairs of zeros, for all the non-trivial solutions of (12), and define

$$P^+ = \{(\zeta_1, \zeta_2) \in P, \, s'(\zeta) > 0\} \quad \text{and} \quad P^- = \{(\zeta_1, \zeta_2) \in P, \, s'(\zeta) < 0\}.$$

Lemma 3.2 *(Separation of zeros) Consider the differential Eq. (12). Suppose we have the following conditions:*

1) $\kappa(t)$ is even and periodic, of period 2τ. Under this condition, s is odd and c is even.
2) s is periodic, of period 2τ, with $s > 0$ on $]0, \tau[$.

Denote by $\alpha > 0$ a value of s. Define

$$S_\alpha = \{t \in \mathbf{R}, \, |s(t)| = \alpha\}.$$

Suppose that α is such that S_α doesn't contain any zero of s'; S_α is then a discrete set. Then:

– *two consecutive zeros in P^+ of any solution of (12) are separated by two distinct elements of S_α if and only if $c(t_0) + c(t_1) \geq 0$, where t_0, t_1 are the smallest positive consecutive reals in S_α.*
– *two consecutive zeros in P^- of any solution of (12) are separated by two distinct elements of S_α if and only if $-c(t_0') - c(t_1') + 2\frac{c'(\tau)}{s'(\tau)}s(t_0') \geq 0$, where t_0', t_1' are the biggest elements $< \tau$ in S_α.*

Proof We start with the following observation: consider two independent solutions of (12) with a_1 and a_2, b_1 and b_2 the two respective zeros in $[-\tau, \tau]$ such that $a_1 < b_1$ and $a_2 < b_2$; Lemma 3.1 ensures the existence of a one-to-one correspondence between $]a_1, b_1[$ and $]a_2, b_2[$ that sends $t \in]a_1, b_1[$ to the unique zero in $]a_2, b_2[$ of the solution (taken up to a multiplicative constant) vanishing at t.

When κ is even, $s_1(t) := -s(-t)$ (resp. $c_1 := c(-t)$) is also a solution of (12) with the same initial conditions as s (resp. c). Hence s is odd and c is even. Now, since we have an action of the operator A on the space V of the solutions of (12), it suffices to show this when $\zeta = 0$ and $\zeta = \tau$. Denote by ϕ_0 and ϕ_1 the two solutions vanishing at $-t_0$ and $-t_1$ respectively, defined by

$$\phi_i(t) = c(t) + \frac{c(t_i)}{s(t_i)}s(t), \quad i = 0, 1.$$

By definition, $s(t_0) = s(t_1)$. An easy computation then gives for $i = 0, 1$:

• $\phi_i(-t_i) = 0$

- $\phi_i(0) = 1$
- $\phi_i(t_{1-i}) = c(t_0) + c(t_1)$.

When $c(t_0) + c(t_1) \geq 0$, $\phi_i(0)$ and $\phi_i(t_{1-i})$ are both non-negative so that ϕ_i doesn't vanish on $[0, t_{1-i}[$ by Lemma 3.1. Denote by z_0 and z_1 the zeros of ϕ_0 and ϕ_1 respectively in $[0, \tau]$; we thus have $z_0 \geq t_1$ and $z_1 \geq t_0$, the last inequalities being reversed when $c(t_0) + c(t_1) < 0$. Hence, using the above observation, we deduce that when $c(t_0) + c(t_1) \geq 0$ is satisfied, two consecutive zeros in $[-\tau, \tau]$ of a solution of the Jacobi equation are always separated by two elements of S_α. Now, suppose that $c(t_0) + c(t_1) < 0$, one can find, using the same observation as above, a solution of (12) vanishing at z_0 and z_1 in $[-\tau, \tau]$, such that $-t_0 < z_0 < 0$ and $0 < z_1 < t_1$.

In the same way, two consecutive zeros in $[0, 2\tau]$ of a solution of the Jacobi equation are separated by two elements of S_α if and only if

$$\psi(2\tau - t_1') + \psi(2\tau - t_0') \geq 0,$$

where ψ is the solution satisfying $\psi(\tau) = 1$, $\psi'(\tau) = 0$. This solution is given by $\psi(t) = \frac{1}{c(\tau)}(c(t) - \frac{c'(\tau)}{s'(\tau)}s(t))$. Now an easy computation gives

$$c(2\tau - t) = c(t) - 2\frac{c'(\tau)}{s'(\tau)}s(t).$$

Indeed, $c(2\tau - t)$ is again a solution of (12): write $c(2\tau - t) = \lambda_1 c(t) + \lambda_2 s(t)$. Setting successively $t = 0$ and $t = \tau$ yields $\lambda_1 = c(2\tau) = 1/s'(2\tau) = 1$, and $\lambda_2 = -2c'(\tau)/s'(\tau)$. Now, observe that $c(\tau) < 0$ for, if not, c would vanish twice in $[0, \tau]$, and this is not possible by lemma 3.1. This ends the proof. □

Notation: Fix $\alpha > 0$ a value of s, and consider an interval I of $\mathbf{R}$. We say that I satisfies the property $(\mathfrak{P})$ if for any two consecutive zeros $z_0 < z_1$ of a solution of (12), such that $z_0 \in I$, z_0 and z_1 are separated by two distinct elements of S_α.

Definition 3.3 We say that an interval $I =]x, y[$ of $\mathbf{R}$ is a domino associated to α if $x, y \in S_\alpha$ and I contains only one element of S_α.

If γ is a geodesic tangent to K at $t = 0$, then $\beta(t) = \beta'(0)s(t)$. So when $\alpha = |C|/|\beta'(0)|$, where C is the Clairaut constant related to γ, points of S_α correspond to the points where $|\beta(t)| = |C|$, which are exactly the points of γ where the norm of K vanishes. So if I is a domino of $\mathbf{R}$, the geodesic restricted to I lies in a domino of the surface, in the sense of Definition 2.1.

Corollary 3.4 *Suppose the conditions of Lemma 3.2 are satisfied, and that in addition, there exist exactly two elements of S_α in $[0, \tau]$, hence between any two zeros of s. Keeping the same notations as in Lemma 3.2, we have:*

- *a domino containing 0 has the property $(\mathfrak{P})$ if and only if $c(t_0) + c(t_1) \geq 0$,*
- *a domino containing τ has the property $(\mathfrak{P})$ if and only if $-c(t_0) - c(t_1) + 2\frac{c'(\tau)}{s'(\tau)}s(t_0) \geq 0$.*

Lemma 3.5 *Consider the differential Eq. (12). Suppose the conditions of Lemma 3.2 are satisfied, and that in addition there exist exactly two elements of S_α in $[0, \tau]$. Suppose further that κ is τ periodic and s is τ antiperiodic, i.e.*

$$s(\tau + t) = -s(t), \forall t \in \mathbf{R}.$$

Then a domino containing 0 has the property $(\mathfrak{P})$ if and only if s realizes the minimum distance between the zeros of the solutions of the Eq. (12).

Proof An easy computation gives

$$c(\tau - t) = 2\frac{c(\tau/2)}{s(\tau/2)}s(t) - c(t) \quad \forall t \in \mathbf{R}. \tag{13}$$

In this case, we have $t_1 = \tau - t_0$ so that $c(t_0) + c(t_1) \geq 0$ reads $c(\tau/2) \geq 0$.
Let z_0 and z_1 be two any consecutive zeros of a solution of Eq. (12). If ϕ is a solution of (12), $\phi(. + \tau)$ is also a solution, so one can assume that $z_0 \in]-\tau, 0[$. Write

$$\phi(t) = c(t) - \frac{c(z_0)}{s(z_0)}s(t),$$

we have $\phi(z_0) = 0$, $\phi(0) = 1$, and $\phi(z_0 + \tau) = -2\frac{c(\tau/2)}{s(\tau/2)}s(z_0)$, by formula (13) above.
We see that $\phi(0)$ and $\phi(z_0 + \tau)$ have the same sign if and only if $c(\tau/2) \geq 0$. In this case, ϕ doesn't vanish on $[0, z_0 + \tau[$ by Lemma 3.1, and $z_1 \geq z_0 + \tau$. This ends the proof. □

Lemma 3 in [3] gives a result on the zeros of the solutions of the differential Eq.

$$u''(x) + p(x)u(x) = 0, \tag{14}$$

under certain restrictive assumptions on the function $p(x)$. Under these assumptions, the even solution of the equation realizes the minimum distance between such zeros. We shall adapt the lemma to the case where the coefficient is a periodic function; the same proof works in this case.

Lemma 3.6 *Let $p(x)$ have the following properties:*

(a) p is continuous;
(b) $p(-x) = p(x)$;
(c) the even solution of the differential Eq. (14) vanishes for $\alpha > 0$ and does not vanish for $-\alpha < x < \alpha$;
(d) $p(x)$ is non-increasing for $0 < x < \alpha$;
(e) p is periodic, with period 2α.

Let α_1, α_2 be any pair of consecutive zeros of any non-trivial solution of (14). Then

$$\alpha_2 - \alpha_1 \geq 2\alpha.$$

Note: in [3], the condition (d) reads "$p(x)$ is non-increasing for $0 < x < +\infty$". When $p(x)$ is assumed to be periodic of period 2α, it is enough to suppose this property on $]0, \alpha[$.

4 Conjugate points of Lorentzian tori with a Killing vector field

4.1 Conjugate points from the Killing vector field

Given a Lorentzian torus with Killing field K, modeled on E_f^u, we have the following result from [2]:

Theorem 4.1 *(Theorem 5.29 [2]). Let $f \in C^\infty(\mathbf{R}, \mathbf{R})$ be a non-constant periodic function, and let T be a torus modeled on E_f^u. If T has no conjugate points, then:*

(1) the set of connected components of $\{f \neq 0\}$ is locally finite,
(2) f changes sign between bands with a common boundary,

(3) f' changes sign once in a band,
(4) each component defines a type II band in the torus.

   A critical orbit of K is an orbit corresponding to a critical point of $\langle K, K \rangle$. These leaves are geodesics (they are exactly the set of leaves of K on which $\nabla_K K = 0$).

Lemma 4.2 *On a torus such that condition (3) of Theorem 4.1 is satisfied, the critical orbits of K are without conjugate points.*

Proof The curvature is given by $\kappa = f''(x)/2$ in the x-coordinate. From hypothesis (3), it follows that a local maximum of f is necessarily positive and a local minimum is negative. This yields the following:

Fact 4.3 *The timelike critical orbit of K corresponding to a local minimum of f lies in a region of non-negative curvature, and the spacelike critical orbit (corresponding to a local maximum of f) lies in a region of non-positive curvature. On the other critical orbits of K, we have $\kappa = 0$.*

   It is then straightforward that Eq. (12) does not have a solution vanishing twice on such geodesics.   □

   As pointed out in the introduction of this paper, we restrict our attention to non-null geodesics. The geodesics perpendicular to K are already known to be without conjugate points in the torus (see [1, Proposition 1.8]); this also holds for the geodesics such that β does not vanish, by Lemma 3.1. We are left with the case in which γ is not perpendicular to K and β vanishes without being identically zero. We begin by investigating the conjugate points produced by β.

   The geodesics are always thought of as being in the extension E_f^u.

Lemma 4.4 *Let γ be a non-null geodesic not perpendicular to K^u. Assume γ is not a critical geodesic of K^u. If f satisfies conditions (2)-(3) of Theorem 4.1 above, we have:*

  (i) *β does not vanish more than one time in a band.*
 (ii) *β vanishes if and only if γ crosses a type III band in E_f^u.*
(iii) *a geodesic γ tangent to K^u in the torus leaves the square in which β vanishes and crosses the type I neighboring band in E_f^u. So if the torus contains only type II bands, the geodesic lies in $\widetilde{T}$ exactly in a domino.*

  Proof (i) Let p be a point where γ is tangent to K^u in the torus, and denote by B the band containing p. Define $U := \{z \in B, z \text{ is an extremum of } \langle K^u, K^u \rangle\}$; condition (3) on f implies that this set is connected, hence splits the band into two connected components. In the interior of the band, K has type ϵ, so the function $\epsilon \langle K, K \rangle$ is positive on B. Suppose β vanishes another time in B. It appears from the formula $\epsilon \langle K^u, K^u \rangle = C^2 - \beta^2$ that p is a local maximum of $\epsilon \langle K^u, K^u \rangle$ on γ, so if q is the closest point to p in which β vanishes, p and q are necessarily in the same connected component, and in addition, $\epsilon \langle K^u, K^u \rangle$ is decreasing from p to q. Now since $\langle K^u, K^u \rangle(p) = \langle K^u, K^u \rangle(q) = \epsilon C^2$, this means that $\epsilon \langle K^u, K^u \rangle$ is constant between the two points, thus γ is a critical orbit of K^u, contrary to our assumption.
  (ii) Let p be a point where γ is tangent to K^u, and denote by B the square containing p. If γ remains in B, it asymptotically approaches an orbit of K^u contained in B on which the norm of K^u is the same as on the point where γ is tangent to K^u (see Lemma

2.16); the same argument above shows that this is impossible. It follows that γ leaves the square by crossing a null leaf of K^u on the boundary of B. Now, denote by l_0 the leaf of the foliation orthogonal to K^u containing p. The generic reflection that fixes l_0 is an isometry of the band that preserves $\gamma_{|B}$. By Remark 2.5, γ crosses a type III band containing the interior of B. The converse is given by Lemma 2.19 (ii).

(iii) By (ii), γ leaves the square in E_f^u in which β vanishes by crossing a type III band. Now, since f changes sign between two consecutive bands, γ cannot cross the type III neighboring band. Repeated application of Lemma 2.16 shows that γ cannot remain in that band either. In addition, γ cannot cross a type II band unless it is perpendicular to K^u. So γ crosses the type I neighboring band. Now, since f changes sign between two consecutive bands, γ cannot cross the type III neighboring band. Repeated application of Lemma 2.16 shows that γ cannot remain in that band either. In addition, γ cannot cross a type II band unless it is perpendicular to K^u. So γ crosses the type I neighboring band.

$\square$

Corollary 4.5 *If (T, K) is a torus satisfying the hypotheses in Theorem 4.1, then the Killing vector field doesn't produce conjugate points in the torus, meaning that the Jacobi field defined by its orthogonal component does not vanish twice in the torus.*

Proof Let γ be a non-null geodesic not perpendicular to K, lifted and then extended to E_f^u, such that β vanishes at least twice. We claim that γ crosses a type I band between any two consecutive zeros of β. Indeed, by Lemma 4.4 (i), if β vanishes in some band of the torus, then β has only one zero in that band. Besides, γ leaves the band and crosses the type I neighboring band (Lemma 4.4 (iii)), and it follows from the point (ii) of the lemma that β does not vanish while γ crosses the type I band. This proves the claim. In consequence, when the torus is assumed to have only type II bands, these zeros are never both in the torus. $\square$

4.2 Invariant geodesics and conjugate points

We begin by stating a lemma that will be used at the end of this section. Choose an orientation on the torus, and let $p \in T$ be a point such that $\langle K(p), K(p) \rangle = 0$. If the orbit of K containing p belongs to the first line of the null cone bordering the negative cone, fix a null vector field L in the maximal ribbon of $\widetilde{T}$ containing p such that $\langle L, K \rangle = -1$; otherwise, take L such that $\langle L, K \rangle = 1$. This way, we get a local coordinate x in the ribbon containing p, which we globalize to E_f^u in a unique way.

Lemma 4.6 *Let γ be a geodesic in E_f^u, then*

$$\epsilon \beta'(t) = \frac{1}{2} f'(x(\gamma(t)).$$

Proof We can assume that the orbit of K containing p belongs to the first line of the null cone bordering the negative cone. In the ribbon containing p, call it U, set $K = CT + \beta N$ and $L = \lambda_1 T + \lambda_2 N$; λ_1 and λ_2 are never vanishing functions on γ. In fact, the choice of L leads to $\lambda_1 = -\epsilon \lambda_2$. We have $\nabla_T K = \beta' N$ and $\nabla_N K = \beta' T$. An easy computation then gives $\nabla_L K = \epsilon \beta' L$, yielding

$$\frac{1}{2} \nabla_L \langle K, K \rangle = \epsilon \beta'.$$

Now in the ribbon obtained from U by a generic reflection, the local coordinate is defined by the null vector field $L^{'}$ such that $\langle L^{'}, K \rangle = 1$, and $L^{'}$ belongs to the other null line field. We can check that this gives the same formula, finishing the proof.   □

Corollary 4.7 *Let (T, K) be a torus modeled on E_f^u, with f a non-constant periodic function such that $f^{'}$ changes sign once in a band. Then a non-null geodesic γ that lies in a band provided $|t|$ is large enough, asymptotically approaches a critical orbit of K^u, and this orbit is either timelike or spacelike, depending on the type of γ.*

Proof Provided $|t|$ is large enough, γ is transverse to K (see Lemma 4.4, (i)), so an application of Lemma 2.16 shows that γ asymptotically approaches a leaf of K^u whose type is the same as γ's, i.e. $x^{'}$ goes to 0 on γ, hence

$$\lim_{t \to \infty} \beta(t) = 0, \tag{15}$$

since $x^{'}(t)^2 = \beta(t)^2$. On the other hand, $\epsilon \beta^{'}(t)$ converges by Lemma 4.6 above; combining this with (15), we see that this limit is necessarily zero.   □

Let γ be a non-null geodesic such that β vanishes without being identically zero, para-matrized so that $\beta(0) = 0$.

Lemma 4.8 *Assume β vanishes twice and let ω be the half-distance between two consecutive zeros (measured in the parameter t of the geodesic). Then β is odd and periodic, of period 4ω (in particular, the distance between two consecutive zeros of β is always 2ω). Furthermore, κ is a 4ω-periodic and even function.*

Proof The proof is based on a powerful geometric ingredient of Lorentzian tori with a Killing vector field: the generic reflection that fixes a non-degenerate leaf of $K^{\perp}$, defined on the saturation of the leaf by the flow of K, extends to a global isometry of the extension (see Proposition 2.10). A point where β vanishes is either a saddle point or corresponds to a point where γ is tangent to K, depending on whether the geodesic is orthogonal to K or not. The behavior being slightly different in each case, we choose to consider them separately. Suppose γ is not a leaf of $K^{\perp}$; the reflection that fixes the geodesic perpendicular to K that passes through $\gamma(0)$ is an isometry that preserves γ (actually, it reverses the orientation), sending K to $-K$; it sends $\gamma(t)$ to $\gamma(-t)$ yielding for all t, $\beta(-t) = -\beta(t)$. Composing two reflections that correspond to two consecutive zeros of β gives an isometry preserving γ, which is actually a translation of the parameter t by 4ω. It sends $\gamma(t)$ to $\gamma(t + 4\omega)$, and gives for all t,

$$\beta(t + 4\omega) = \beta(t),$$

as expected. Now suppose that γ is perpendicular to K; β vanishes at the saddle points. Let p be a saddle point on γ. According to [2], we might choose two generic reflections each one fixing a leaf of $K^{\perp}$ passing through p, such that their composition is the reflection with respect to p. It is not hard to see that the obtained reflection is an isometry preserving γ; it again reverses its orientation and sends K to $-K$. Composing two such reflections as before (corresponding to two saddle points), the same conclusions follow.
The same arguments provide the proof of the statement on κ.   □

Definition 4.9 A geodesic of E_f^u with β a non-zero periodic function is called "β-periodic" or an "invariant geodesic", for they are invariant by an isometry of the extension.

The term "β-periodic" will be adopted below because it seems to be more suggestive of the type of resulting Jacobi equation. We simply say "periodic geodesic" instead of "β-periodic".

If we suppose that f has simple zeros, the geodesics perpendicular to K are all periodic.

Corollary 4.10 *Let γ be a non-null geodesic tangent to K in the torus; we distinguish two different behaviors of γ in E_f^u: either γ is periodic, or it asymptotically approaches a critical orbit of K with infinite t; in this case, the geodesic is tangent to K only one time.*

Proof If γ cuts a null orbit of K^u denoted by $\mathfrak{l}$, we claim that γ crosses a type I band B containing $\mathfrak{l}$ if and only if $C^2 > \sup_B \epsilon \langle K^u, K^u \rangle$. Indeed, if γ crosses a type I band, it is everywhere transverse to K^u in that band, for if β vanishes, either γ remains in the band or it crosses a type III band containing $\mathfrak{l}$. Writing $C^2 - \beta^2 = \epsilon \langle K^u, K^u \rangle$, we see that $C^2 > \sup_B \epsilon \langle K^u, K^u \rangle$. Now, assume this inequality holds, then the same argument as in the proof of Lemma 4.4, (iii), shows that γ crosses the type I band B. In fact, as long as this inequality holds, γ remains in the ribbon containing $\mathfrak{l}$, by cutting the leaves of K^u transversally.

Now, if γ enters a band where C^2 is reached by $\epsilon \langle K^u, K^u \rangle$ in that band, using Corollary 4.7 and the previous lemma, we distinguish two different cases:

a) C^2 is not a critical value of f; in this case, γ meets tangentially the closest orbit of K^u to the boundary of the band meeting γ, on which $C^2 = \epsilon \langle K^u, K^u \rangle$, so that β vanishes a second time. The geodesic is thus periodic.

b) C^2 is a critical value of f, which amounts to saying that the corresponding orbit of K^u is a geodesic. In this case, if the closest orbit of K^u to the boundary meeting γ is not a geodesic, then γ behaves as in a); otherwise, it approaches the orbit asymptotically, with an infinite t.

□

Remark 4.11 This corollary holds for every torus modeled on E_f^u such that f is a non-constant periodic function. When assumptions (2)-(3) in Theorem 4.1 hold, the periodic geodesics leave every maximal ribbon in E_f^u, by crossing a type I band between two zeros of β.

Definition 4.12 Define $\mathcal{L}_K^*(T)$ as the subset of $\mathcal{L}_K(T)$ such that the function f induced by the norm of K has simple zeros and satisfies conditions (3)–(4) of Theorem 4.1. In particular, the assumption "f has simple zeros" implies conditions (1)–(2) of Theorem 4.1.

Given a torus in $\mathcal{L}_K^*(T)$, using the results in Sect. 3, we shall give a necessary and sufficient condition for geodesics (other than the critical orbits of K) tangent to K in the torus to be without conjugate points in it.

Recall that when assumptions (2)–(3) in Theorem 4.1 hold, if γ is a geodesic tangent to K at $p \in T$, and not a critical orbit of K, then p is the unique point of T where γ is tangent to K.

Proposition 4.13 *(Characterization of geodesics without conjugate points) Let (T, K) be a torus modeled on E_f^u, with f satisfying the properties in Theorem 4.1. Let γ be a geodesic tangent to K in the torus; assume γ is not a critical orbit of K and set $t = 0$ at this point. Then γ is without conjugate points in the torus if and only if $c(t_0) + c(t_1) \geq 0$, where t_0, t_1 are the smallest positive consecutive reals such that the norm of K vanishes on γ.*

Proof The two reals t_0 and t_1 do exist by Lemma 4.4. The geodesic lies in the torus in a domino, either for $t \in [-t_1, t_0]$ or for $t \in [-t_0, t_1]$, t_0 and t_1 being as in the statement above. Assume γ is periodic; if $c(t_0) + c(t_1) \geq 0$, the intervals $[-t_1, t_0]$ and $[-t_0, t_1]$ satisfy the

property ($\mathfrak{P}$) for $\alpha = |C|$, by Lemma 3.2. All we have to show is that this implies that γ is without conjugate points in the torus. Setting $\alpha = |C|$, S_α is the set of points in which the norm of K vanishes. Assume ($\mathfrak{P}$) is satisfied on $[-t_1, t_0]$, i.e. two zeros of any solution of (12) in $[-t_1, t_0]$ (or in $[-t_0, t_1]$) are separated by two zeros of $\langle K, K \rangle$. Since γ lies in T^2 in a domino, it contains only one zero of $\langle K, K \rangle$ in the torus. Therefore, two zeros of a solution of (12) cannot be both in the torus. Now, suppose that $c(t_0) + c(t_1) < 0$; one can find by Lemma 3.2 a Jacobi field along γ vanishing at z_0 and z_1 in $[-2\omega, 2\omega]$, such that $-t_0 < z_0 < 0$ and $0 < z_1 < t_1$.

The case in which γ asymptotically approaches a critical orbit of K works in much the same way by proving, following Lemma 3.2, that the property ($\mathfrak{P}$) is again equivalent to having $c(t_0) + c(t_1) \geq 0$. The detailed verification is left to the reader. $\qquad\square$

Theorem 4.14 *Let (T, K) be a torus in $\mathcal{L}_K^*(T)$ modeled on E_f^u. Denote by $(x_n)_{n\in\mathbf{Z}}$ the sequence of zeros of f, the x_n's being taken in increasing order. If the torus has no conjugate points, then*

$$f'(x_n) + f'(x_{n+1}) = 0, \quad \forall n \in \mathbf{Z}.$$

Proof Denote again by $(0), .., (\mathfrak{n})$ the distinct bands of the torus, (i) and $(i + 1)$ having a common boundary and opposite signs. Denote by $x_i < x_{i+1}$ the zeros of f in the band (i). Let γ be a geodesic tangent to K in a band (i) of the torus. Set $t = 0$ at this point. For $|C|$ small enough, the geodesic crosses one type I band before β vanishes again in a copy of the band $(i + 2)$ of the ribbon containing (i). The torus contains an isometric image of the geodesic γ restricted to a domino containing the zero of β in (i), and an isometric image of the geodesic restricted to a domino containing the zero of β in $(i + 2)$. By Corollary 3.4 and Lemma 4.13, these geodesics are without conjugate points in the torus if and only if $c(t_0) + c(t_1) \geq 0$ and $-c(t_0) - c(t_1) + 2\frac{c'(2\omega)}{s'(2\omega)}s(t_0) \geq 0$. Consider a sequence of geodesics γ_n approaching a non-degenerate geodesic γ_∞ perpendicular to K. Then, $|C|$ tends to 0, $t_0^n \to 0$, and $t_1^n \to 2\omega_\infty$, where ω_∞ is associated to γ_∞. We admit that $c(t_0) + c(t_1)$ depends continuously on γ (this will be proved in Sect. 5). Letting $|C|$ go to 0, we have

$$c_n(t_0^n) + c_n(t_1^n) \to c_\infty(0) + c_\infty(2\omega_\infty)$$

$$\text{and} \quad -c_n(t_0^n) - c(t_1^n) + 2\frac{c_n'(2\omega_n)}{s_n'(2\omega_n)}s_n(t_0^n) \to -c_\infty(0) - c_\infty(2\omega_\infty),$$

where the solution c_∞ associated to γ_∞ is then evaluated at two consecutive saddle points of γ_∞. Denote them by p_i and p_{i+1}. On the other hand,

$$c_\infty(0) + c_\infty(2\omega) = \frac{1}{s_\infty'(0)} + \frac{1}{s_\infty'(2\omega)} = \frac{\beta_\infty'(p_i) + \beta_\infty'(p_{i+1})}{\beta_\infty'(p_{i+1})}.$$

We see that if the torus is without conjugate points, $c_\infty(0) + c_\infty(2\omega)$, and then $\beta_\infty'(p_i) + \beta_\infty'(p_{i+1})$ should be zero. Using Lemma 4.6, this yields $f'(x_i) + f'(x_{i+1}) = 0$, for all i. $\square$

Remark 4.15 The geodesic parametrization of a null orbit of K is incomplete if and only if it corresponds to a simple zero of f.

Remark 4.16 When the geodesic is incomplete, it is easy to see that a geodesic parametrization is given by $e^{-\frac{1}{2}\eta f' t}$, where $K = \partial_t$, and $\eta = \pm 1$ depends on the null orbit of K and the choice of the x-coordinate (once the coordinate is fixed, two consecutive null orbits of K in the torus give (η_1, η_2), such that $\eta_1 + \eta_2 = 0$); therefore, Theorem 4.14 above can be stated in a more geometric way, as in Theorem 1.3 in the introduction of this paper.

Corollary 4.17 *T_{CP} can be obtained as a limit of Lorentzian tori with conjugate points, and admitting a Killing field.*

Proof This is because the necessary condition for the absence of conjugate points, obtained in Theorem 4.14 above is a closed condition. So if we perturbate the function f corresponding to the Clifton–Pohl torus, we get metrics on the torus with a Killing field, arbitrarily close to T_{CP} and containing conjugate points.   □

Remark 4.18 In [1] it is shown that the space of Lorentzian tori without conjugate points is a closed subset of $\mathcal{L}(T)$ endowed with the C^∞ topology, and that the Clifton-Pohl torus is on the boundary of this set. In the proof, the construction of the sequence of such metrics converging to T_{CP} killed the Killing vector field.

4.3 A class of Lorentzian tori without conjugate points

The local geometry of a Lorentzian torus with a Killing vector field is determined by the function f induced by the norm of $\widetilde{K}$ in the x-coordinate of the universal cover. In privileging this point of view, we obtain a family of Lorentzian tori without conjugate points admitting a Killing vector field with pairwise non-isometric universal cover; these examples include the Clifton–Pohl torus and quadratic variations of it.

In our examples, we assume that f has two zeros in its smallest period, and that $\exists a \in \mathbf{R}, \ f(a+t) = f(a-t)$ (in particular, $\mathfrak{n} = 2$; see Definition 2.7 for the definition of $\mathfrak{n}$). We then get additional isometries in E_f^u, known as non-generic reflections in [2], that lead to additional symmetries on the β solution. In this case, given a non-null geodesic γ, not perpendicular to K, the behavior in b), Corollary 4.10, does not appear, and β vanishes (without being identically zero) if and only if the geodesic is periodic. With that assumption on f, κ has two symmetries, one of which about $t = 0$ and the other about $t = \omega$, hence κ is 2ω periodic, and β satisfies the following:

$$\beta(2\omega - t) = \beta(t) \ \ \forall \, t \in \mathbf{R}.$$

So let γ be a periodic geodesic, and set $t = 0$ at the point in the torus where γ is tangent to K. Write $\beta(t) = \beta'(0)s(t)$ where s is as in Sect. 3. Lemma 3.5 together with Proposition 4.13 imply that the absence of conjugate points in the torus for such geodesics is equivalent to the fact that β realizes the minimum distance between the zeros of the solutions of the Jacobi equation. A large class of Lorentzian tori with no conjugate points is given in the following theorem:

Theorem 4.19 *Let f be a periodic function that satisfies the following properties:*

  i) *f has simple zeros,*
 ii) *f' changes sign one time in a band,*
iii) *$f' \cdot f''' \leq 0$,*
 iv) *$\exists a \in \mathbf{R}, \ f(a+t) = f(a-t)$,*
  v) *f has two zeros in the smallest period of f.*

Then, a torus modeled on E_f^u and all of whose bands are of type II has no conjugate points.

Proof Let γ be a non-null geodesic, and assume that β vanishes. Either β is identically zero, in which case the geodesic is a critical orbit of K without conjugate points (Lemma 4.2), or it isn't identically zero, then γ is periodic. The geodesic therefore has no conjugate points in

the torus if and only if β realizes the minimum distance between the zeros of the non-trivial solutions of the Jacobi equation. Put $p(t) = \epsilon\kappa(t + \omega)$ and consider the differential Eq.

$$u''(t) + \epsilon\kappa(t + \omega)u(t) = 0 \tag{16}$$

We have the following:

i) p is even; this is a consequence of κ being even and 2ω periodic. Indeed, write $p(-t) = \epsilon\kappa(-t + \omega) = \epsilon\kappa(t - \omega) = \epsilon\kappa(t + \omega) = p(t)$;

ii) the even solution is given by $\beta(t + \omega)$;

iii) $p(t)$ is decreasing for $0 < t < \omega$.

To see this, denote $x_1 < x_2$ two consecutive critical points of f which correspond to extremal values. Condition (ii) implies that f' does not change sign on $]x_1, x_2[$. Since the curvature is given by $f''(x)/2$ in the x coordinate, the condition $f'.f''' \leq 0$ implies that on a ribbon, κ is either decreasing or increasing between the two orbits of K corresponding to x_1 and x_2, depending on whether f reaches its minimal value on x_1 or x_2. Now the geodesic is transverse to K over $]0, 2\omega[$ and lies in a ribbon of the Lorentzian surface E_f^u. Since $\mathfrak{n} = 2$, the geodesic crosses only one extremal orbit of K between 0 and 2ω, the one corresponding to $t = \omega$. We then get that $\kappa(t)$ is decreasing (resp. increasing) for $\omega < t < 2\omega$ if the geodesic is spacelike (resp. timelike), implying condition (iii).

Given Lemma 3.6, the even solution $\beta(t + \omega)$ then realizes the minimum distance between the zeros of the non-trivial solutions of Eq. (16). Therefore, when the torus has only type II bands, these geodesics have no conjugate points in the torus. This completes the proof of Theorem 4.19. $\qquad\qquad\square$

4.4 Some examples of Lorentzian tori without conjugate points

Definition 4.20 We call a quadratic variation of the Clifton–Pohl torus a metric of the form

$$g = \frac{2dxdy}{Q(x, y)},$$

where Q is a positive definite quadratic form of determinant 1.

Proposition 4.21 *The quadratic variations of the Clifton–Pohl torus are contained in the family obtained in Theorem 4.19.*

Proof Considering $-g$ instead of g if necessary, we can assume that the quadratic forms are given by

$$Q(x, y) = ux^2 + 2vxy + wy^2, \quad \text{with } u > 0, w > 0, uw - v^2 = 1.$$

Write $Q(x, y) = (\sqrt{u}x + \frac{v}{\sqrt{u}}y)^2 + (\frac{1}{\sqrt{u}}y)^2$. Applying the change of variables $x' = \sqrt{u}x, y' = \frac{1}{\sqrt{u}}y$, we reduce to the metrics of the form

$$g = \frac{2dxdy}{x^2 + (y + ax)^2},$$

where $a \in \mathbf{R}$. These metrics admit a Killing vector field given by $K = x\partial_x + y\partial_y$. We have now

$$\kappa = -2K.K - 2a.$$

Hence, f is a solution of the differential Eq.

$$f''(x) + 4f(x) - 4a = 0,$$

and it is given by $f(x) = \sin(2x) - 2a\cos^2(x)$.

The Clifton–Pohl torus and its quadratic variations are then contained in the family obtained before, and are therefore without conjugate points. □

Other examples of Lorentzian tori without conjugate points, belonging to this family, are given at the end of this paper.

5 Deformation of a Lorentzian torus without conjugate points

Recall that a torus in $\mathcal{L}_K(T)$ without conjugate points satisfies conditions (1)-(4) in Theorem 4.1. So we defined $\mathcal{L}_K^*(T)$ to be the subset of Lorentzian tori in $\mathcal{L}_K(T)$ containing only type II bands (condition (4)), and modeled on E_f^u such that f has simple zeros and satisfies the third condition of the theorem (the condition on the zeros implies (1)-(2)). A torus without conjugate points in $\mathcal{L}_K^*(T)$ satisfies in addition the property in Theorem 4.14, which is a kind of "pointwise symmetry" on f; in the sequel we denote by $S\mathcal{L}_K^*(T)$ the set of Lorentzian metrics in $\mathcal{L}_K^*(T)$ satisfying this condition.

The family introduced in Theorem 4.19 gives examples of metrics in $\mathcal{L}_K^*(T)$ without conjugate points; we wish to know if this property is stable by deformation in $S\mathcal{L}_K^*(T)$. Our main concern in this section will be to obtain stability criteria for these metrics, i.e. conditions on f from which conclusions may be drawn as to the stability character of the property.

This section contains 5 Subsections. In Sect. 5.1, we prove that for metrics in $\mathcal{L}_K(T)$, the Killing field depends smoothly on the metric. This allows us to define in Sect. 5.2 a continuous function depending on the metric, that controls the conjugate points. Recall that the universal cover of a Lorentzian torus admitting a Killing field appears as an open subset of a bigger surface which contains conjugate points. For the stability question, a special interest will be given in Sesct. 5.3 and 5.4 to the case in which conjugate points are on the boundary. We will give sufficient conditions on the metric to avoid this situation, and obtain in the last section a stability result.

5.1 On the Killing vector field of a Lorentzian metric on *T*

The space of smooth Lorentzian metrics on a torus is equipped with the C^r topology; since the torus is compact, this space is metrisable. If K is a Killing field for a metric g, then λK, $\lambda \in \mathbf{R}^*$, is also a Killing vector field for g. Actually, if a (non-flat) metric on a torus admits a Killing field, then the latter is unique up to a multiplicative constant; this was proved in [2]. To fix one, one fixes an orientation on the torus (by fixing a volume form ν on it) and a vector field J everywhere transverse to K, together with a point p on T in which $g(K, K)(p) \neq 0$, and takes $\lambda_0 K$ such that $g(\lambda_0 K, \lambda_0 K)(p) = \eta$, $\eta = \pm 1$, and $\nu(K, J) > 0$ (the latter is possible, for the foliation of K is orientable). We simply denote it by K. This determines K on the orbit of K containing p, and on a null geodesic passing through this point (Clairaut's constant). This way, one determines K on the saturation of the geodesic by the flow of K, i.e. on a ribbon; doing it on the ribbons in turn, K is determined on the whole torus. In the following, K is fixed this way.

Now consider a sequence of Lorentzian metrics $g_n \in \mathcal{L}_K(T)$ which converges to a non-flat metric $g \in \mathcal{L}_K(T)$ in the C^r topology, $r \geq 4$; denote by K the Killing vector field of g, K_n

that of g_n for all n. A natural question is to see whether K_n converges to K. For convenience, we start by examining the foliations associated to these vector fields. We want to provide the space of foliations of the torus, denoted by $\mathcal{F}(T)$, with a "natural" topology. The projective bundle over a torus being a trivial bundle, we choose a trivialization and we associate to a foliation its tangent field, which is then a smooth map from T into P^1. One identifies $\mathcal{F}(T)$ with its image in $C^r(T, P^1)$ equipped with the C^r topology, and gets a topology on $\mathcal{F}(T)$; this topology is independent of the choice of the trivialization.

Proposition 5.1 *If g_n is a sequence of Lorentzian metrics in $\mathcal{L}_K(T)$ converging to a non-flat metric $g \in \mathcal{L}_K(T)$ for the C^r topology, $r \geq 4$, then the sequence of foliations associated to K_n converges to that of K, for the C^{r-3} topology. Then, if p is a point on T where $g(K, K)(p) = \eta$, $\eta = \pm 1$, then for n sufficiently large, $K_n(p)$ has type η; fix K_n by setting $g_n(K_n, K_n)(p) = \eta$, such that $K_n(p)$ converges to $K(p)$. The sequence K_n so obtained converges to K for the C^{r-4} topology.*

In the following, $g \in \mathcal{L}_K(T)$ is a non-flat metric; we denote by U the open set in which κ, the sectional curvature associated to g, is a submersion. On this set, κ defines the foliation associated to K. The open set of T not containing the null orbits of K will be denoted by V.

Fact 5.2 *Consider a sequence of metrics $g_n \in \mathcal{L}_K(T)$ converging to g for the C^r topology; then the sequence of foliations associated to K_n converges to that of K for the C^{r-3} topology, on every compact subset of U.*

Proof Observe that the space of submersions over a compact variety M is an open set in the space of C^1 functions over M. Let B be a compact subset of U; provided we take n large enough, κ_n is also a submersion on B that defines the foliation associated to K_n.
In smooth local coordinates (s, t), define

$$X_\kappa(s, t) := \left(-\frac{\partial \kappa}{\partial t}, \frac{\partial \kappa}{\partial s} \right),$$

a vector field associated to κ. We have $d\kappa(X_\kappa) = 0$. Moreover, since K acts isometrically on the torus, the curvature is constant along the leaves of K, so we also have $d\kappa(K) = 0$. Now, since g is not flat, it follows that K and X_κ are colinear, so that X_κ defines the foliation of K. It follows that if κ_n converges to κ for the C^{k-2} topology, the foliations they define converge for the C^{k-3} topology. Note that the curvature defines the foliation but not the Killing field; indeed, if X is such that $d\kappa(X) = 0$, then $d\kappa(\phi X) = 0$ for any function ϕ on the torus. $\qquad\square$

Fact 5.3 *Denote by $\mathcal{F}$ (resp. $\mathcal{F}_n$) the foliation associated to K (resp. K_n). Suppose there exists $q \in V$ such that $K_n(q)$ converges to $K(q)$. Fix K and K_n, for n big enough, by setting $g(K, K)(q) = g_n(K_n, K_n)(q) = \eta$, $\eta = \pm 1$. If $\mathcal{F}_n$ converges to $\mathcal{F}$ on V for the C^k topology, with $k \leq r$, then K_n converges to K on every compact subset of the connected component of q contained in V. The convergence holds for the C^k topology.*

Proof Fix a Riemannian metric S on T, and let X (resp. X_n) be a vector field on T tangent to $\mathcal{F}$ (resp. $\mathcal{F}_n$), such that $X(q)$ (resp. $X_n(q)$ is $\mathbf{R}^+$-collinear to $K(q)$ (resp. $K_n(q)$), and $S(X, X) = 1$ (resp. $S(X_n, X_n) = 1$). Since the foliations are orientable, the oriented foliations still converge, so that X_n converges to X for the C^k topology. Next, we define $\bar{K}$ (resp. $\bar{K}_n$) to be the vector field on the connected component of V containing q, $\mathbf{R}^+$-collinear to X (resp. X_n) such that $g(\bar{K}, \bar{K}) = \epsilon = g_n(\bar{K}_n, \bar{K}_n)$, for n big enough. We have $\bar{K}_n \xrightarrow{C^k} \bar{K}$. Write $K = h\bar{K}$ and $K_n = h_n \bar{K}_n$, where $h, h_n > 0$, for all n. This yields $h = \sqrt{\epsilon g(K, K)}$, which implies that h (resp. h_n) is invariant by the flow of K (resp. K_n).

Now, write

$$g(\nabla_X K, X) = 0, \quad \forall X.$$

This yields the following Eq. on h

$$(\nabla_X h) g(\bar{K}, X) + h\, g(\nabla_X \bar{K}, X) = 0, \quad \forall X. \tag{17}$$

Let $c(s)$ be a curve containing q and transverse to K, and define a C^r-diffeomorphism ϕ by setting $\phi(s, t) = F(t, c(s))$, where F is the flow of $\bar{K}$. There exists an open neighborhood of $\mathbf{R}^2$ on which the flows of $\bar{K}_n$ are all defined, for n big enough. So define in the same way a diffeomorphism ϕ_n by $\phi_n(s, t) = F_n(t, c(s))$, where F_n is the flow of $\bar{K}_n$.
Since $\bar{K}_n$ is C^k close to $\bar{K}$, for n large enough, ϕ_n is C^k close to ϕ. Set $X = \frac{\partial\phi}{\partial s}$ and $X_n = \frac{\partial\phi_n}{\partial s}$ in Eq. (17) above; combining this with the fact that h is invariant by $\bar{K}$, $h \circ \phi$, written in the (s, t) coordinates, satisfies the following differential Eqs.:

$$\frac{\partial(h \circ \phi)}{\partial t} = 0,$$
$$\frac{\partial(h \circ \phi)}{\partial s} g(\bar{K}, X) \circ \phi + (h \circ \phi) g(\nabla_X \bar{K}, X) \circ \phi = 0.$$

This yields $h \circ \phi(s, t) = h \circ \phi(s, 0)$ and $h_n \circ \phi_n(s, t) = h_n \circ \phi_n(s, 0)$, for all t. The continuity of solutions in the initial conditions and the coefficients of the equation permits now to assert that $h_n \circ \phi_n(s, 0)$ is C^{k-1} close to $h \circ \phi(s, 0)$, hence $h_n \circ \phi_n$ is C^{k-1} close to $h \circ \phi$, for n big enough. Since $\phi_n \xrightarrow{C^k} \phi$, one easily shows that actually h_n converges to h on a neighborhood of q, for the C^{k-1} topology, and then on any compact subset of the connected component of V containing q. Now, look at Eq. (17), the C^{k-1} convergence of h_n and the C^k convergence of $\bar{K}_n$ imply the C^{k-1} convergence of the first order derivatives of h_n, hence the C^k convergence of h_n. □

Fact 5.4 *Let g_n be a sequence of metrics in $\mathcal{L}_K(T)$ converging to g for the C^r topology; K (resp. K_n) the Killing field of g (resp. g_n) fixed as in the previous fact. Denote by R a maximal ribbon in $\widetilde{T}$, and let L be the null vector field on it defined by $g(L, K) = 1$; define L_n (for n big enough) to be the null vector field on R such that $g_n(K_n, L_n) = 1$. The C^k convergence of K_n to K on R, with $k \leq r$, leads to the C^k convergence of L_n to L.*

Proof Let $\bar{L}$ (resp. $\bar{L}_n$) be the null vector field on R, $\mathbf{R}^+$-collinear to L (resp. L_n), such that $S(\bar{L}_n, \bar{L}_n) = S(\bar{L}, \bar{L}) = 1$, where S is a Riemannian metric on T. By definition, the sequence of (oriented) foliations associated to L_n converges to that of L, so that $\bar{L}_n \xrightarrow{C^r} \bar{L}$. Write $L = s\bar{L}$ and $L_n = s_n\bar{L}_n$, s and s_n being non-vanishing functions. This yields

$$g(\bar{L}, K) = \frac{1}{s} \quad \text{and} \quad g_n(\bar{L}_n, K_n) = \frac{1}{s_n}.$$

It follows that if K is C^k close to K, then s_n is C^k close to s, which finishes the proof. □

Fact 5.5 *Let g_n be a sequence of metrics in $\mathcal{L}_K(T)$ converging to g for the C^r topology; take $q \in V \cap U$ and fix K and K_n as in Fact 5.3. Denote by R a maximal ribbon for the metric g, containing q, and define L and L_n on R as in the previous fact, then L_n is C^{r-3} close to L on any compact subset of R.*

Proof Combining Fact 5.2 and Fact 5.3, we can say that L_n converges to L for the C^{r-3} topology on any compact subset of the connected component of q in $V \cap U$. Denote by B a

set saturated by K, where this convergence holds. Now, in a local chart, the geodesics of g are the integral curves of the vector field of $TT(T)$ defined by

$$Z = (Z^1, Z^2) = \left(y^k, -\sum_{i,j} \Gamma^k_{i,j}(x) y^i y^j \right).$$

The components of Z involve the first derivatives of g, so if g_n converges to g for the C^r topology, then Z_n converges to Z for the C^{r-1} topology. Therefore, there exists an open neighborhood $\mathcal{U}$ of the zero section of $\mathbf{R} \times T(T)$ on which the flows Φ_n of Z_n are all defined, for all n but a finite number. Furthermore, the sequence Φ_n converges to Φ on any compact subset of $\mathcal{U}$ (for the C^{r-1} topology). Let $p \in R$; for an appropriate $t_0 > 0$, Φ_{t_0} is a diffeomorphism from an open set $T\mathcal{V}$ of $T(T)$, where $\mathcal{V}$ is an open set in B, into an open set $T\mathcal{W}$, where $\mathcal{W}$ is a neighborhood of p. The sequence of diffeomorphisms $\Phi_n(t_0, .)$ converges to $\Phi(t_0, .)$; combining this with the fact that L_n tends to L on $\mathcal{V}$, we get $\Phi_n(t_0, L_n) \xrightarrow{C^{r-3}} \Phi(t_0, L)$. Now write $\Phi_n(t_0, L_n) = L_n \circ \Phi_n$ and $\Phi(t_0, L) = L \circ \Phi$. It follows that $L_n \circ \Phi_n \xrightarrow{C^{r-3}} L \circ \Phi$, hence $L_n \xrightarrow{C^{r-3}} L$ in the neighborhood of p.   □

We go back to the proof of Proposition 5.1.

Proof We start with the following observation.

Observation: Suppose g has non-constant curvature κ, and consider $p \in T$ such that $\alpha := \kappa(p)$ is a regular value of κ and $g(K, K)(p) \neq 0$; set $g(K, K)(p) = \eta$, $\eta = \pm 1$, taking $\lambda_0 K$ if necessary. The set $\kappa^{-1}(\alpha)$ of isolated orbits of K contains the orbit of K containing p. Now let $\{\psi^t\}$ be a flow of K. Since the stages ψ^t of the flow are isometries, the exponential map commutes with the flow; we write

$$exp_{\psi^t(p)} o \, \psi^t_* = \psi^t \circ exp_p, \tag{18}$$

This geometric ingredient allows, once an orbit of K is given, to get all the orbits of K. Indeed, fix $q \in T$; p and q can be joined by a curve made of (broken) null geodesics, cutting each orbit of K transversally; call this curve c. Denote by $\gamma(t)$ the integral curve of K containing p; put $\gamma(0) = p$, and $g(\dot{\gamma}(p), \dot{\gamma}(p)) = 1$. Let L denote the null vector field along γ, tangent to the null foliation containing c, and such that $\langle L, K \rangle = 1$. Now (18) gives:

$$\psi^t(q) = exp_{\gamma(t)}(s L_{\gamma(t)}),$$

where $exp_p(s L(p)) = q$, defining the integral curve of K containing q.

According to Fact 5.2 and Fact 5.3, $K_n \xrightarrow{C^{r-3}} K$ on every compact subset of the connected component of p in $V \cap U$. Denote by B an open set containing p where this convergence holds. Furthermore, in every ribbon containing p, the sequence L_n of null vector fields defined in a ribbon by $g_n(L_n, K_n) = 1$, for n large enough, converges to the null vector field L defined in that ribbon by $g(L, K) = 1$.

Now, we denote by Exp (resp. Exp_n) the exponential map of g (resp. g_n). Let q be a point in a ribbon containing p, and let $t_0 > 0$ such that $q' = Exp(t_0 L(q)) \in B$. There exists a neighborhood $\mathcal{W}$ of the zero-section of a subset of $T(T)$, such that $\mathcal{W} = \{tv, t \in [0, t_0], v \in \mathcal{U}\}$, where $\mathcal{U}$ is a neighborhood of $L(q)$ in $T(T)$, in which the exponential maps of the metrics g_n are defined for all n but a finite number. Furthermore, Exp_n converges C^{k-1} to Exp on every compact subset of $\mathcal{W}$. Now, according to the previous observation, if $z \in \pi(\mathcal{U})$, where $\pi : T(T) \to T$ is the natural projection, the integral curve of K (resp. K_n) containing z is

given by ψ^t (resp. ψ_n^t), such that:

$$\psi^t(z) = Exp_{\psi^t(\phi(z))}(-t_0 L(\psi^t \circ \phi(z)))), \quad \text{where} \ \phi(z) = Exp_z(t_0 L(z)) \in B.$$

Since L_n converges C^{r-3} to L, and the flows of K_n converge C^{r-3} to the flow of K in B, then ψ_n^t converges C^{r-3} to ψ^t, leading to the C^{r-4} convergence of K_n to K on $\pi(\mathcal{U})$. The result on the torus follows from the fact that any point on it can be brought to a point in B by a broken null geodesic everywhere transverse to K.   □

Corollary 5.6 *Let (T, K) be a torus in $\mathcal{L}_K^*(T)$; the number of bands, their signs and types are preserved in a neighborhood of the torus in $\mathcal{L}_K(T)$.*

Proof For the number of bands and their signs, this is an easy consequence of the previous proposition. For the type preservation, recall that the null-leaves of K on a band of type II belong to different null-foliations, while on a band of type I, they belong to the same null-foliation. Since we cannot have a type III band in the torus, the result follows.   □

Corollary 5.7 *Let $g \in \mathcal{L}_K(T)$ a non-flat metric with Killing field K, and let $g_n \in \mathcal{L}_K(T)$ be a sequence of metrics, with Killing field K_n, converging to g for the C^r topology, $r \geq 4$. Let R be a maximal ribbon in $\widetilde{T}$ for the metric g, and fix $p \in R$. Define $\phi : R \to I \times \mathbf{R}, \phi(q) = (x(q), y(q))$, with $\phi(p) = (0, 0)$; define in the same way ϕ_n on R satisfying $\phi_n(p) = (0, 0)$. Then, ϕ_n converges C^{r-4} to ϕ on every compact subset of R.*

Proof Proposition 5.1 implies the C^{r-4} convergence of the Killing fields, hence that of the transverse coordinates, using Lemma 2.11. Now, define a volume form ν on R by setting $\nu(K, L) = 1$, where L is the null vector field on R satisfying $g(L, K) = 1$. The coordinate $y \in C^\infty(R, \mathbf{R})$ can be defined by $i_{\widetilde{L}}\nu = dy$ and $y(p) = 0$. In the same way, consider for n sufficiently large a volume form ν_n on R defined by $\nu_n(K_n, L_n) = 1$, where L_n is the null vector field on R satisfying $g_n(L_n, K_n) = 1$. The coordinate $y_n \in C^\infty(R, \mathbf{R})$ is given by $i_{L_n}\nu_n = dy_n$, $y_n(p) = 0$. The convergence of ϕ_n to ϕ follows from Proposition 5.1 and Fact 5.4.   □

Remark 5.8 The application ϕ (resp. ϕ_n) defined above is a diffeomorphism on R (resp. on every saturated subset of R). Take an open set J of I, contained in I_n for all n but a finite number. The pullback of the metrics by these applications read $(\phi^{-1})_*g = 2dxdy + f(x)dy^2$ and $(\phi_n^{-1})_*g_n = 2dxdy + f_n(x)dy^2$, $(x, y) \in I \times \mathbf{R}$, whose extensions to $\mathbf{R}^2$ that give the maximal ribbons in E_f^u and $E_{f_n}^u$ respectively, are proved to be convergent in Lemma 2.11.

Lemma 5.9 *Let (U, K), $I = I \times \mathbf{R}$ be a Lorentzian domino where the unique null orbit of K (represented by $x = 0$) is incomplete. Denote by g the metric on U and let (g_n, K_n) be a sequence of metrics on U such that $(g_n, K_n) \xrightarrow{C^r} (g, K), r \geq 2$. Then, there exists a neighborhood J of 0 in which the coordinate neighborhoods for g_n given by Eq. (5) converges C^{r-2} to that of g.*

Proof The neighborhood J is given by Lemma 2.12, and the convergence follows from the previous lemma, and Eqs. (5) and (8), [2], that define these coordinates.   □

5.2 A continuous function controlling conjugate points

Define

$$\Omega = \{(g, p) \in \mathcal{L}_K^*(T) \times T, \ g(K, K)(p) \neq 0 \ \text{and} \ \nabla_K K(p) \neq 0\}.$$

For $g \in \mathcal{L}_K^*(T)$, the points $p \in T$ such that $(g, p) \in \Omega$ are the points of T which are neither on a null orbit of K nor on a critical orbit. Take $(g, p) \in \Omega$ and consider the geodesic γ_p in T tangent to K at p. Set $t = 0$ at p and define $c(t_0) + c(t_1)$ as in the previous section. Define a function $\mathcal{Z}$ on the open set Ω as follows:

$$\mathcal{Z} : \Omega \longrightarrow \mathbf{R}$$

$$\mathcal{Z}(g, p) = \frac{c(t_0) + c(t_1)}{s(t_0)}$$

where t_0, t_1 and c are exactly as in the previous section.

Lemma 5.10 *A metric g in $\mathcal{L}_K^*(T)$ is without conjugate points if and only if $\mathcal{Z}(g, p) \geq 0$, $\forall p \in T$ such that $(g, p) \in \Omega$.*

Proof This follows from Lemma 4.2 and Proposition 4.13. □

Proposition 5.11 *If the space of Lorentzian metrics on T is equipped with the C^2 topology, then $\mathcal{Z}$ is continuous on Ω.*

Proof Let $(g_n, p_n) \rightarrow (g, p)$ in Ω, where g_n converges to g for the C^2 topology. In $\mathbf{R}^2$ equipped with the metric $2dxdy + f(x)dy^2$ (resp. $2dxdy + f_n(x)dy^2$), let γ (resp. γ_n) be the geodesic tangent to $\phi_* K$ (resp. $\phi_{n_*} K_n$) at $\phi(p)$ (resp. $\phi_n(p_n)$), where $\phi, \phi_n, n \in \mathbf{N}$, are the local charts defined in Corollary 5.7, satisfying $\phi(p) = \phi_n(p) = 0$. The geodesic sequence γ_n converges uniformly to γ on any compact set of $\mathbf{R}^2$.
Let us rewrite the Jacobi equations along γ_n and γ:

$$u''(t) + \epsilon \kappa_n(t) u(t) = 0 \tag{19}$$

$$u''(t) + \epsilon \kappa(t) u(t) = 0 \tag{20}$$

κ_n converges uniformly to κ on any compact set. The pairs of solutions (s, c) and (s_n, c_n) are given by the same initial conditions so s_n and c_n converge uniformly to s and c respectively, on any compact set containing γ.
Lemma 4.4 states that γ restricted to the subset of $\mathbf{R}^2$ contained in $\widetilde{T}$ lies in a domino. This holds also for γ_n, for n large enough. Denote by x_0 (resp. x_0^n) the zero of f (resp. f_n) in it, and set $x = z_0$ (resp. $x = z_0^n$) at p (resp. p_n). We have by use of Fact 2.13 that x_0^n converges to x_0. Write

$$t_0 = \int_{z_0}^{x_0} \frac{1}{\sqrt{C^2 - \epsilon f(x)}} dx.$$

This yields $t_0^n \rightarrow t_0$, hence the convergence of $c_n(t_0^n)$ to $c(t_0)$. To show that $c_n(t_1^n)$ tends to $c(t_1)$, we also show that t_1^n tends to t_1. Since a geodesic crosses a type I band between t_0 and t_1, we can write

$$t_1 = t_0 + \int_{x_0}^{x_1} \frac{1}{\sqrt{C^2 - \epsilon f(x)}} dx.$$

This yields $t_1^n \rightarrow t_1$ in the same way, finishing the proof. □

Now assume that $T \in S\mathcal{L}_K^*(T)$, i.e. the function f induced by the norm of K satisfies the property

$$f'(x_n) + f'(x_{n+1}) = 0, \ \forall n \in \mathbf{Z},$$

where $(x_n)_{n \in \mathbf{Z}}$ is the sequence of zeros of f, taken in an increasing order. Then, using $cs' - c's = 1$, one can check that the $\mathcal{Z}$ function can be written in the following way:

$$\mathcal{Z} : \Omega \longrightarrow \mathbf{R}$$

$$\mathcal{Z}(g, p) = \frac{c'(t_0) - c'(t_1)}{s'(t_0)}$$

Proposition 5.12 *The functions $\mathcal{Z}$ can be extended to the set*

$$\widetilde{\Omega} = \{(g, p) \in S\mathcal{L}_K^*(T) \times T, \ g(K, K)(p) \geq 0 \ \text{and} \ \nabla_K K(p) \neq 0\}$$
$$\amalg \{(g, p) \in S\mathcal{L}_K^*(T) \times T, \ g(K, K)(p) \leq 0 \ \text{and} \ \nabla_K K(p) \neq 0\},$$

into a continuous function.

Proof Take $(g, p) \in S\mathcal{L}_K^*(T) \times T$ such that $g(K, K)(p) = 0$. The null orbit of K containing p splits the domino containing it into two connected components (interior of bands), one timelike band and one spacelike band. Take $(p_n)_n$ a sequence of points in the torus converging to p, and contained in the timelike band. The sequence $(\gamma_n)_n$ of timelike geodesics tangent to K at p_n, $\forall n$, converges to the null orbit of K containing p. The function $\mathcal{Z}(g)$ being constant along the leaves of the Killing field for each metric g, we don't change the behavior of $\mathcal{Z}$ if we move the γ_n's by the flow of K in such a way that the sequence converges to a timelike geodesic γ_∞ orthogonal to K and crossing the spacelike neighboring band. We can do this by letting p_n go to the saddle point of the null-orbit of K containing p, along a leaf orthogonal to K contained in B. This way, we see that $\mathcal{Z}(g, p_n)$ converges to $-c'_\infty(2\omega)$, where s_∞ and c_∞ are the solutions of the Jacobi equation on γ_∞, with initial conditions at the saddle point. This limit does not depend on the choice of γ_∞; denote it by $\mathcal{Z}(g, p)^-$. In the same way, if γ_n's are spacelike, we get another limit using this time a spacelike geodesic orthogonal to K and passing through the same saddle point. Now, fix a metric in $S\mathcal{L}_K^*(T)$ and denote by $(B_i)_{i \in I}$ the bands of the torus for the metric g; the boundary of a band B_i is made of two null orbits of K, call them $\partial_1 B$ and $\partial_2 B$. Define for $p \in \amalg_{i \in I} B_i$:

$$\widetilde{\mathcal{Z}}(g, p) = \begin{cases} \mathcal{Z}(g, p) & \text{if } (g, p) \in \mathring{B}_i, \\ \mathcal{Z}(g, p)_1^\epsilon & \text{if } (g, p) \in \partial_1 B, \\ \mathcal{Z}(g, p)_2^\epsilon & \text{if } (g, p) \in \partial_2 B, \end{cases}$$

for all $i \in I$, where $\epsilon = \pm 1$, depending on the type of the band.
This gives a well defined function $\widetilde{\mathcal{Z}}$ on the set

$$\widetilde{\Omega} = \{(g, p) \in S\mathcal{L}_K^*(T) \times T, \ g(K, K)(p) \geq 0 \ \text{and} \ \nabla_K K(p) \neq 0\}$$
$$\amalg \{(g, p) \in S\mathcal{L}_K^*(T) \times T, \ g(K, K)(p) \leq 0 \ \text{and} \ \nabla_K K(p) \neq 0\},$$

Now, to prove that $\widetilde{\mathcal{Z}}$ is continuous, take $(g, p) \in S\mathcal{L}_K^*(T) \times T$ such that $g(K, K)(p) = 0$, and consider a sequence of metrics $g_n \in S\mathcal{L}_K^*(T)$ converging to g. Lemma 5.9 and Lemma 2.12 allow us to consider a coordinate neighborhood centered in 0 and given by Eq. (5), in which the saddle points of g_n and that of g are represented by the origin. Take a sequence γ_n of timelike geodesics converging to the null orbit of K containing p. Denote by p_n the points where γ_n is tangent to K_n. One can find a sequence of points q_n such that $\widetilde{\mathcal{Z}}(g_n, q_n) = \widetilde{\mathcal{Z}}(g_n, p_n)$, for all $n \in \mathbf{N}$, and the sequence of geodesics σ_n tangent to K_n at q_n converges to a timelike geodesic σ_∞ orthogonal to K. We can achieve this in the following way: there exists a sequence $(t_n)_n$ such that the sequence of points $F^{t_n}(p_n)$, where F^t is the flow of K, converges to the origin. We use the convergence of the flows and define $q_n := F_n^{t_n}(p_n)$,

where F_n^t is the flow of K_n (we choose the points q_n to vary on a leaf orthogonal to K). This way, we obtain $c'_n(t_0^n) \to c'_\infty(0) = 0$ and $s'_n(t_0^n) \to s'_\infty(0) = 1$, where s_∞ and c_∞ are the solutions of the Jacobi equation on γ_∞, with initial conditions at the origin.

Now, we want to prove that $c'_n(t_1^n)$ converges to $c'_\infty(2\omega_\infty)$. Take an interval I in which σ_n converges to σ_∞. Take $\tau_0 \in I$; σ_n and σ_∞ may be seen as solutions of the differential equation which gives locally the geodesics of g_n and g, with initial conditions given by $\sigma_n(\tau_0)$ and $\sigma'_\infty(\tau_0)$ respectively. Let us consider a coordinate neighborhood U, given by Lemma 2.12, containing the saddle point that belongs to the null orbit of the type I neighboring band containing $\sigma'_\infty(\tau_0)$. Choosing τ_0 so that $\sigma_\infty(\tau_0), \sigma_n(\tau_0) \in U$ for n large enough, we can assert that σ_n (extended to U) converges to σ_∞ in U. The result follows.   □

From now on, the function $\widetilde{\mathcal{Z}}$ will be simply denoted by $\mathcal{Z}$.

5.3 Digression: Jacobi equations all of whose solutions are periodic

In what follows, g belongs to the family given in Theorem 4.19.

Lemma 5.13 *If a metric $g \in \mathcal{L}_K(T)$ belongs to the family given in Theorem 4.19, then $\mathcal{Z}$ vanishes on (g, p), $p \in T$, if and only if the Jacobi equation corresponding to γ_g, the g-geodesic tangent to K at p, has only periodic solutions.*

Proof For such a metric, $\mathcal{Z}$ is defined for $p \in T$ such that γ_g is an invariant geodesic, and the additional isometries of g given by condition (iv) in Theorem 4.19 imply that $\kappa(t)$ is 2ω periodic and that β is 2ω anti-periodic. In this case, formula (13) in Lemma 3.5 holds, so that $\mathcal{Z}$ vanishes if and only if $c(\omega) = 0$, which is equivalent to saying that c is 2ω anti-periodic (hence 4ω periodic).   □

Remark 5.14 An equation

$$u'' + r(t)u = 0, \ t \in \mathbf{R},$$

where the coefficient $r(t)$ is T-periodic is called a Hill equation. When all the solutions are assumed to be T-antiperiodic, with one zero on $[0, T[$, the following inequality holds (see [4], Appendix B p. 230)

$$T \int_0^T r(t)dt \le \pi^2,$$

with equality only for constant $r(t)$.

Let $p, r : [a, b] \to \mathbf{R}$, where $p(x) > 0$, and consider the Eq.

$$(py')' + \lambda r y = 0, a \le x \le b, \tag{21}$$

with boundary conditions

$$y(a) = y(b) = 0. \tag{22}$$

According to [11] (p. 288), there are two infinite sequences of parameter values $0 \le \lambda_1 \le \lambda_2...$, $0 \ge \lambda_{-1} \ge \lambda_{-2}...$, each one of which has $+\infty$ and $-\infty$ for its only point of accumulation, and for each parameter λ_m (resp. λ_{-m}), a solution y_m (resp. y_{-m}) satisfying (22) exists. The number of zeros of y_m in $[a, b]$ is $m + 1$.

Let γ be an invariant geodesic for the metric g, and consider the differential system

$$\begin{cases} u'' + \lambda \epsilon \kappa u = 0, & t \in [0, 2\omega] \\ u(0) = u(2\omega) = 0, \end{cases} \tag{23}$$

where κ, the sectional curvature along γ_g, is (see Sect. 4.3) symmetric with respect to $t = 0$ and $t = \omega$. Denote by λ_1 the least positive eigenvalue of the differential system (23), which exists by the statement above, and let y_1 be the corresponding eigenfunction. Since κ is symmetric with respect to $t = \omega$, an easy computation gives either $y_1'(\omega) = 0$ or $y_1(\omega) = 0$. Furthermore, y_1 does not vanish on $]0, 2\omega[$, so actually $y_1'(\omega) = 0$. It follows that y_1 is a solution of the new system

$$\begin{cases} u'' + \lambda \epsilon \kappa u = 0, & t \in [0, \omega] \\ u(0) = u'(\omega) = 0. \end{cases} \tag{24}$$

Now, if λ is an eigenvalue of the system (24), with corresponding eigenfunction u defined on $[0, \omega]$, then, by reflecting u about the line $t = \omega$, one gets a solution of the differential system (23), for κ is symmetric with respect to ω.

This proves that λ_1 is also the least positive eigenvalue of the differential system (24).

The following fact follows from the proof of Lemma 3, [3].

Fact 5.15 *Suppose that the coefficient r in Eq. (21) is symmetric with respect to $t_0 = \frac{a+b}{2}$. If there exists a solution y_0 of $(py')' + ry = 0$ satisfying $y_0(a) = y_0(b) = 0$, symmetric with respect to t_0, with y_0 not vanishing on $]0, 2\omega[$, then the least positive eigenvalue of the system (21)-(22) is $\lambda_1 = 1$.*

Lemma 5.16 *When g belongs to the family given in Theorem 4.19, the following inequality holds*

$$\int_0^{2\omega} \epsilon \kappa(t) dt \le \frac{\pi^2}{2\omega}.$$

for all invariant geodesics.

Proof Let λ_1 be the least positive eigenvalue of the system

$$\begin{cases} u'' + \lambda \epsilon \kappa u = 0, & t \in I = [0, \omega] \\ u(0) = u'(\omega) = 0. \end{cases} \tag{25}$$

We have the following inequality from Theorem 6, [5]

$$\lambda_1 \le \frac{(\pi/2)^2}{d^2 \omega^2}, \quad \text{where } d^2 = \min_{t \in I} \frac{\int_t^\omega \epsilon \kappa(t) dt}{\omega - t}.$$

The function β is a solution of the Jacobi equation that satisfies

$$\beta(0) = \beta'(\omega) = 0, \quad \beta' > 0 \text{ on } [0, \omega[,$$

This implies using Fact 5.15 above that the least positive eigenvalue λ_1 of the system (25) is $\lambda_1 = 1$. Now set

$$h(t) = \frac{1}{\omega - t} \int_t^\omega \epsilon \kappa(t) dt, \quad t \in I$$

We have

$$h'(t) = \frac{1}{\omega - t} \left[\frac{1}{\omega - t} \int_t^\omega \epsilon \kappa(t) dt - \epsilon \kappa(t) \right].$$

Using the mean value theorem, we show the existence of $\tau \in]t, \omega[$, such that $\frac{1}{\omega-t}\int_t^\omega \epsilon\kappa(t)dt = \epsilon\kappa(\tau)$. This gives

$$h'(t) = \frac{1}{\omega-t}(\epsilon\kappa(\tau) - \epsilon\kappa(t)), \quad \tau \in]t, \omega[.$$

Condition (iii) in Theorem 4.19 implies that $\epsilon k(t)$ is an increasing function on $[0, \omega]$, so that

$$\min_{t\in I} \frac{\int_t^\omega \epsilon\kappa(t)dt}{\omega - t} = \frac{1}{\omega}\int_0^\omega \epsilon\kappa(t)dt.$$

$\square$

Lemma 5.17 *Consider a metric in the family given in Theorem 4.19; suppose that the Jacobi equation of a geodesic γ has only periodic solutions, then*

$$\sup\left\{\frac{1}{\omega}\int_\omega^{2\omega} \epsilon\kappa(t)dt, \; -\epsilon\kappa(0)\right\} \geq \frac{\pi^2}{4\omega^2}. \tag{26}$$

It follows that either $-\epsilon\kappa(0) \geq \frac{\pi^2}{4\omega^2}$ or κ is constant along γ.

Proof Let λ_1 be the least positive eigenvalue of the system

$$\begin{cases} u'' + \lambda\epsilon\kappa u = 0, & t \in J = [0, \omega] \\ u(\omega) = u'(2\omega) = 0. \end{cases} \tag{27}$$

The following inequality follows from Theorem 5, [5]:

$$\lambda_1 \geq \frac{(\pi/2)^2}{D^2\omega^2}, \quad \text{where} \quad \max_{t\in I} \frac{|\int_t^{2\omega} \epsilon\kappa(t)dt|}{2\omega - t} = D^2.$$

Suppose that the Jacobi equation has only periodic solutions, i.e. $c(\omega) = 0$. In this case, we have $c(\omega) = c(3\omega) = 0$, and $c(4\omega - t) = c(t)$. Thus, c and the coefficient κ are symmetric with respect to $t = 2\omega$, with c not vanishing on $]\omega, 3\omega[$. It follows that $\lambda_1 = 1$, by use of Fact 5.15 again.

Now set as in the proof of the previous lemma

$$h(t) = \frac{1}{2\omega - t}\left|\int_t^{2\omega} \epsilon\kappa(t)dt\right|, \quad t \in I$$

Let $b \in I$ the smallest real such that $\int_b^{2\omega} \epsilon\kappa(t)dt = 0$. Doing the same computation as before, we show the existence of $\tau \in]t, 2\omega[$ such that

$$h'(t) = \begin{cases} \frac{1}{2\omega-t}\left(\epsilon\kappa(\tau) - \epsilon\kappa(t)\right), & t \in [\omega, b] \\ \frac{1}{2\omega-t}\left(-\epsilon\kappa(\tau) + \epsilon\kappa(t)\right), & t \in [b, 2\omega] \end{cases}$$

Condition (iii) in Theorem 4.19 implies that $\epsilon\kappa(t)$ is a decreasing function on $[\omega, 2\omega]$, so that

$$\max_{t\in I} \frac{|\int_t^{2\omega} \epsilon\kappa(t)dt|}{2\omega - t} = \sup\left\{\frac{1}{\omega}\int_\omega^{2\omega} \epsilon\kappa(t)dt, \; -\epsilon\kappa(0)\right\}.$$

The above inequality reads

$$\sup\left\{\frac{1}{\omega}\int_\omega^{2\omega} \epsilon\kappa(t)dt, \; -\epsilon\kappa(0)\right\} \geq \frac{\pi^2}{4\omega^2}.$$

The last assertion is an easy consequence of Lemma 5.16 and Remark 5.14. $\square$

Corollary 5.18 *Suppose that the curvature has a constant sign on an invariant geodesic γ without being constant, then the Jacobi equation admits a non-periodic solution.*

Proof A geodesic of type ϵ cuts the critical orbit of K of type $-\epsilon$, corresponding to $t = \omega$, on which the curvature is either zero or has sign ϵ; since κ doesn't change sign on γ, $-\epsilon\kappa(0) \leq 0$. Therefore, if the Jacobi equation has only periodic solutions, κ is constant on γ by use of the previous lemma. $\qquad\square$

Lemma 5.19 *Suppose that the curvature vanishes exactly twice on the smallest period of f, and let γ be a geodesic on which $\kappa(t) := \kappa \circ \gamma(t)$ changes sign. If the Jacobi equation associated to γ admits only periodic solutions, then*

$$(\omega - \tau)^2 \epsilon\kappa(\omega) \geq \frac{\pi^2}{4}, \tag{28}$$

where $0 < \tau < \omega$ is the smallest positive real such that $\kappa(\tau) = 0$ on γ.

Proof By assumption, $c(\omega) = 0$. Let $b > 0$ be the smallest positive real such that $c'(b) = 0$. If we assume κ to vanish twice in the period of f, then the zeros of κ are on both sides of a critical orbit of K corresponding to an extremum of f. Indeed, since $\kappa(t)$ is symmetric with respect to $t = 0$ and $t = \omega$, either both zeros of it are in the same band or both of them are on the critical orbits of K where f' changes sign. The latter implies that κ is everywhere positive or everywhere negative on the torus, so it cannot actually happen, unless the torus is flat. In addition, κ changes sign while vanishing. It follows that κ has sign $-\epsilon$ on $[0, \tau[$ and ϵ on $]\tau, \omega]$, so that c is convex on $[0, \tau]$ and concave on $[\tau, \omega]$, hence $\tau < b < \omega$. The curvature does not vanish on $[b, \omega]$; on this interval it is easy to check that the differential Eq.

$$\left(\frac{1}{\kappa}y'\right)' + \epsilon y = 0$$

is satisfied by u', where u is a solution of the Jacobi equation. Let λ_1 be the least positive eigenvalue of the system

$$\left(\frac{1}{\kappa}y'\right)' + \lambda\epsilon y = 0, \qquad t \in I = [b, \omega],$$

$$y(b) = y'(\omega) = 0.$$

The assumption on c leads to $c''(\omega) = 0$, so we have $c''(\omega) = c'(b) = 0$, and c'' does not vanish on $[b, \omega[$; it follows that $\lambda_1 = 1$, as in the proof of the previous lemma.
We have (see Theorem 5, [5])

$$\max_{t \in I} \frac{\omega - t}{\int_t^\omega \epsilon\kappa(t)dt} \geq \frac{\pi^2}{4(\int_b^\omega \epsilon\kappa(t)dt)^2}. \tag{29}$$

Now set

$$h(t) = \frac{\omega - t}{\int_t^\omega \epsilon\kappa(t)dt}, \quad t \in [b, \omega].$$

The same argument as in the previous lemmas shows the existence of $\theta \in]t, \omega[$, such that

$$h'(t) = (\omega - t)(-\epsilon\kappa(\theta) + \epsilon\kappa(t))\frac{1}{(\int_t^\omega \epsilon\kappa(t)dt)^2},$$

so that h is a decreasing function on $[b, \omega]$, and (29) reads

$$(\omega - b) \int_b^\omega \epsilon \kappa(t)dt \geq \frac{\pi^2}{4}.$$

Combining this with the fact that $\epsilon \kappa(t)$ reaches its maximum at $t = \omega$, and that $b > \tau$, we get the desired inequality:

$$(\omega - \tau)^2 \epsilon \kappa(\omega) \geq \frac{\pi^2}{4}.$$

$\square$

5.4 Application: invariant geodesics with non-periodic Jacobi vector field

In what follows, $f \in C^\infty(\mathbf{R}, \mathbf{R})$ belongs to the family given in Theorem 4.19.

Proposition 5.20 *Suppose κ vanishes twice on the smallest period of f. If there exists an invariant geodesic γ_0 on which κ changes sign, such that*

$$-\epsilon f''(x_0) \left(\int_{x_0}^{x_1} \frac{dx}{\sqrt{M_\epsilon - \epsilon f(x)}} \right)^2 < 2\pi^2, \tag{30}$$

where x_0, x_1 are the coordinates of two consecutive points where γ_0 is tangent to K, and $M_\epsilon = \sup \epsilon f(x)$, then, either κ is constant on the band where ϵf is negative, or for all the geodesics such that $C^2 \leq C_0^2 = \epsilon f(x_0)$, where C_0 is the Clairaut constant of γ_0, the Jacobi equation admits a non-periodic solution.

Proof Let $z_0 = 0$ and d be two consecutive zeros of f, such that z_0 and d border the part of f of sign $-\epsilon$. Denote by x_{cr}^ϵ the coordinate of a zero of f', such that either $x_0 \in [x_{cr}^\epsilon, 0]$ or $x_0 \in [d, x_{cr}^\epsilon]$; for simplicity, assume $x_0 \in [x_{cr}^\epsilon, 0]$. We have $x_1 = d - x_0$. Define

$$h(x) = -\epsilon \frac{f''(x)}{2} - \frac{\pi^2}{\left(\int_x^{d-x} \frac{dt}{\sqrt{M_\epsilon - \epsilon f(t)}} \right)^2},$$

where $x \in J = [x_0, 0]$; h is a derivable function since for $x \in J$, $M_\epsilon - \epsilon f(t) > 0$ for every $t \in [x, d-x]$.
A simple computation gives

$$h'(x) = -\epsilon \frac{f^{(3)}(x)}{2} - \frac{2\pi^2}{\sqrt{M_\epsilon - \epsilon f(x)} \left(\int_x^{d-x} \frac{dt}{\sqrt{M_\epsilon - \epsilon f(t)}} \right)^3}.$$

Multiply both sides by $f'(x)$; this gives

$$f'(x)h'(x) = -\epsilon \frac{f^{(3)}(x)f'(x)}{2} - D^2 f'(x),$$

where $D^2 = \dfrac{2\pi^2}{\sqrt{M_\epsilon - \epsilon f(x)} \left(\int_x^{d-x} \frac{dt}{\sqrt{M_\epsilon - \epsilon f(t)}} \right)^3}$.

By assumption, we have $f' f^{(3)} \leq 0$; furthermore, f' has sign $-\epsilon$ on J; combining these two facts, we deduce that h is a decreasing function on J. Now, for $C^2 \leq C_0^2$ i.e. $x \in [x_0, 0]$,

we have

$$h(x_0) \geq -\epsilon \frac{f''(x)}{2} - \frac{\pi^2}{(\int_x^{d-x} \frac{dt}{\sqrt{C_0^2 - \epsilon f(t)}})^2},$$

so if we suppose that (30) is true, then

$$-\epsilon \frac{f''(x)}{2} - \frac{\pi^2}{\left(\int_x^{d-x} \frac{dt}{\sqrt{C^2 - \epsilon f(t)}} \right)^2} < 0, \quad \forall x \in J.$$

Now, recall that $\kappa(t) = f''(x(\gamma(t)))/2$, and $t(d - x) - t(x) = \int_x^{d-x} \frac{1}{\sqrt{C^2 - \epsilon f(x)}} dx$. So the latter inequality is equivalent to $-\epsilon \kappa(0) < \frac{\pi^2}{4\omega^2}$ on γ_C, which ends the proof by Lemma 5.17. $\square$

Proposition 5.21 *Suppose κ vanishes twice on the smallest period of f. Assume there exists an invariant geodesic γ_0 on which κ changes sign, such that*

$$\epsilon f''(x_{cr}^{-\epsilon}) \left(\int_{\zeta_0}^{\zeta_1} \frac{dx}{\sqrt{\epsilon f(x_0) - \epsilon f(x)}} \right)^2 < 2\pi^2, \tag{31}$$

where

  i) *$x_{cr}^{-\epsilon}$ is a critical point of f corresponding to the critical orbit of K of type $-\epsilon$;*
  ii) *$\zeta_0 < x_{cr}^{-\epsilon} < \zeta_1$ are the coordinates of two consecutive points where f'' vanishes, from both sides of $x_{cr}^{-\epsilon}$;*
  iii) *x_0 the point where γ_0 is tangent to K.*

Then for all the geodesics with $C_0^2 = \epsilon f(x_0) \leq C^2 < M_\epsilon$, where C_0 is the Clairaut constant of γ_0, the Jacobi equation admits a non-periodic solution.

Proof Inequality (31) implies that $(\omega_0 - \tau_0)^2 \epsilon \kappa(\omega_0) < \frac{\pi^2}{4}$ on γ_0, where τ_0 is the same as in Lemma 5.19, hence the Jacobi equation on γ_0 admits a non-periodic solution, by use of Lemma 5.19.
This conclusion holds for $C^2 \geq C_0^2$, for $(\omega_C - \tau_C)$ and $\epsilon \kappa(\omega_C)$ (we put C in index to say that it depends on the geodesic γ_C) are decreasing functions of C. $\square$

5.5 Stability by small deformation

Lemma 5.22 *Let $g \in \mathcal{L}_K^*(T)$. Assume κ vanishes $\mathfrak{n}$ times on the smallest period of f, hence twice between any two critical orbits of the Killing field of same type ($\mathfrak{n}$ is the number of zeros of f in a period), and that these zeros are not on the extremums of f. Assume in addition that the zeros of κ are simple zeros. Let γ_∞ be a critical orbit of K corresponding to an extremum of f. Then there exists a neighborhood V of the set $\{(g, \gamma_\infty(t), t \in \mathbf{R})\}$ in $\mathcal{L}_K(T) \times T$, where $\mathcal{L}_K(T)$ is equipped with the C^∞ topology, such that $\mathcal{Z} > 0$ on $V \cap \Omega$.*

Proof This amounts to saying that the geodesics near γ_∞, for metrics close enough to g, where g is the metric on the torus, are without conjugate points. Call B the band containing γ_∞, and fix p a point on γ_∞; it is sufficient to prove that there exists a neighborhood of (g, p) in which $\mathcal{Z}$ is positive; the conclusion will follow from the compactness of γ_∞. So let (g_n, p_n) be a sequence in Ω converging to (g, p). For all n, denote by γ_n the g_n-geodesic

tangent to K_n at p_n. Since $(g_n, p_n) \in \Omega$, we have $\forall n$, $C_n^2 < \sup_B \epsilon g_n(K_n, K_n)$. Choose p to be the origin on T and denote by x (resp. x_n) the transverse coordinate associated to g (resp. g_n), with the fixed origin. We can suppose that the points p_n are all on the same side of the critical orbit of K_n close to γ_∞. Since the zeros of κ are simple zeros, the curvature vanishes twice between two critical orbits of K_n of same type for metrics sufficiently close to g. Set $x = x_0, x = x_1$ (resp. x_0^n, x_1^n) the smallest positive reals such that $g(K, K)$ (resp. $g_n(K_n, K_n)$) vanishes, and $x = \zeta_0, x = \zeta_1$ (resp. ζ_0^n, ζ_1^n) the smallest positive reals such that κ (resp. κ_n) vanishes. When C^2 varies in $I_\lambda = [\sup(\epsilon f(\zeta_0), \epsilon f(\zeta_1)) + \lambda, \sup_B \epsilon g(K, K)]$, $\lambda > 0$ small enough, the g-geodesic γ_C cuts the orbits $x = \zeta_0$ and $x = \zeta_1$ of K. Now $f_n(\zeta_{0,1}^n)$ converges to $f(\zeta_{0,1})$, and $\sup_B \epsilon g_n(K_n, K_n)$ converges to $\sup_B \epsilon g(K, K)$; this ensures the existence of $\Lambda > 0$ such that for n big enough, we have $C_n^2 > \Lambda + \sup(\epsilon f_n(\zeta_0^n), \epsilon f_n(\zeta_1^n))$. Set $t = 0$ at the point where γ_n is tangent to K_n in the torus, and denote by t_0^n, t_1^n the smallest positive reals such that $\beta_n^2(t_0^n) = \beta_n^2(t_1^n) = C_n^2$, and $t_{\zeta_0}^n, t_{\zeta_1}^n$ the smallest positive reals such that $\kappa_n(t_{\zeta_0}^n) = \kappa_n(t_{\zeta_1}^n) = 0$. There are 4 cases, listed in figures 3, 4, 5 and 6.

We first prove that the two sequences $(t_{\zeta_1}^n - t_{\zeta_0}^n)_n$ and $(t_1^n - t_0^n)_n$ are bounded. For this purpose, write

$$t_{\zeta_1}^n - t_{\zeta_0}^n = \int_{\zeta_0^n}^{\zeta_1^n} \frac{1}{\sqrt{C_n^2 - \epsilon f_n(x_n)}} \, dx_n.$$
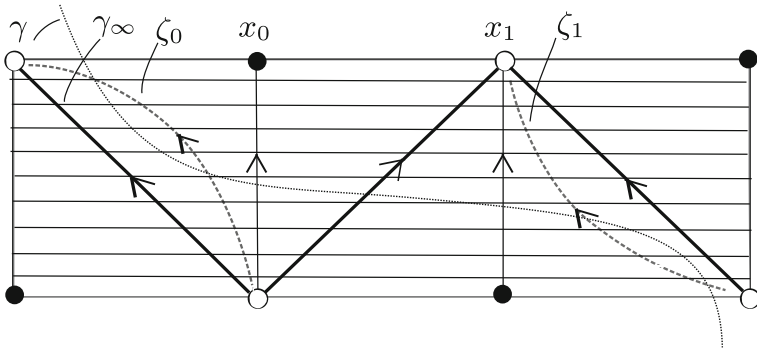

Fig. 3 $\zeta_0 < x_0 < x_1 < \zeta_1$, i.e. $t_{\zeta_0}^n < t_0^n < \omega_n < t_1^n < t_{\zeta_1}^n$
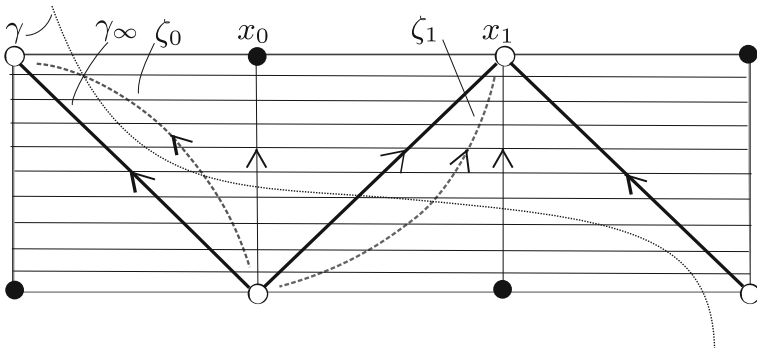

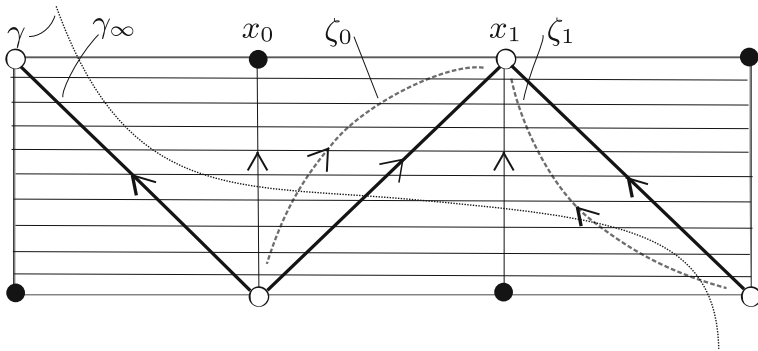**Fig. 4** $\zeta_0 < x_0 < \zeta_1 < x_1$, i.e. $t_{\zeta_0}^n < t_0^n < \omega_n < t_{\zeta_1}^n < t_1^n$

Fig. 5 $x_0 < \zeta_0 < x_1 < \zeta_1$, i.e. $t_0^n < t_{\zeta_0}^n < \omega_n < t_1^n < t_{\zeta_1}^n$
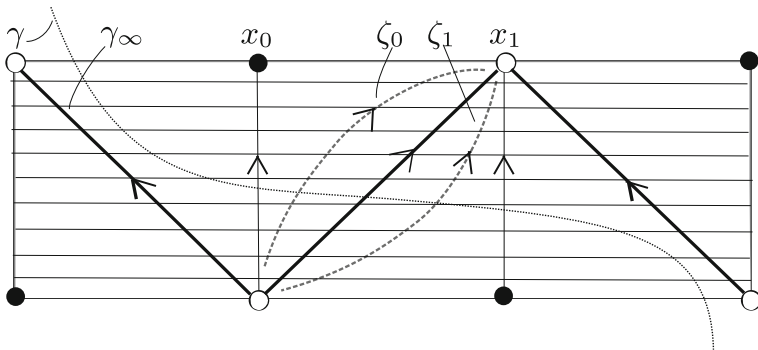

Fig. 6 $x_0 \le \zeta_0 < \zeta_1 \le x_1$, i.e. $t_0^n \le t_{\zeta_0}^n < \omega_n < t_{\zeta_1}^n \le t_1^n$

Let $\mathcal{N}$ be a neighborhood of (g, p) in $\mathcal{L}_K(T) \mathrm{x} T$ in which we have $C_n^2 > \Lambda + \sup(\epsilon f_n(\zeta_0^n), \epsilon f_n(\zeta_1^n))$. For all $x \in [\zeta_0, \zeta_1]$, we have

$$C^2 - \epsilon f(x) \ge \inf\{C^2 - \epsilon f(\zeta_0), C^2 - \epsilon f(\zeta_1)\} = C^2 - \sup\{\epsilon f(\zeta_0), \epsilon f(\zeta_1)\},$$

hence, for metrics in $\mathcal{N}$, we get

$$t_{\zeta_1}^n - t_{\zeta_0}^n = \int_{\zeta_0^n}^{\zeta_1^n} \frac{1}{\sqrt{C_n^2 - \epsilon f_n(x_n)}} \, dx_n \le \int_{\zeta_0^n}^{\zeta_1^n} \frac{1}{\sqrt{C_n^2 - \sup(\epsilon f_n(\zeta_0^n), \epsilon f_n(\zeta_1^n))}} \, dx_n$$

$$\le (\zeta_1^n - \zeta_0^n) \frac{1}{\sqrt{\Lambda}}.$$

Restricting to a subset of $\mathcal{N}$ if necessary, we can find a constant $A > 0$ such that $t_{\zeta_1}^n - t_{\zeta_0}^n \le A$ on this subset. It follows that when n tends to infinity, the difference $t_{\zeta_0^1} - t_{\zeta_0^n}$ is bounded, even if $t_{\zeta_0^n}$ and $t_{\zeta_1^n}$ go to infinity. Similarly, write

$$t_1^n - t_0^n = \int_{x_0^n}^{x_1^n} \frac{1}{\sqrt{C_n^2 - \epsilon f_n(x_n)}} \, dx_n \le (x_1^n - x_0^n) \frac{1}{\sqrt{C_n^2}}.$$

Using that C_n is bounded away from zero establishes that $t_1^n - t_0^n$ is bounded, which is our assertion.

Our next claim is that $c_n(t_{\zeta_1^n}) > 0$, provided n is large enough. The solution c_n is convex for $t \in [0, t_{\zeta_0^n}]$. Denote by τ_n and θ_n the smallest positive reals such that $c_n(\tau_n) = 0$ and $c_n'(\theta_n) = 0$. Suppose the claim were false. Then we could find a sequence $(n_i)_i \to \infty$ such that $\forall i$, $\tau_{n_i} \le \zeta_1^{n_i}$. We have $\beta_{n_i}(t) = \beta_{n_i}'(0) s_{n_i}(t)$; write $\epsilon \beta'(t) = \frac{1}{2} f'(x(\gamma(t)))$, it appears that $\beta_{n_i}'(0) \to 0$ when $i \to \infty$. Using $\langle K_n, K_n \rangle = \epsilon(C_n^2 - \beta_n^2)$, we get

$$s_{n_i}(t_{\zeta_0^{n_i}}) = \frac{C_{n_i}^2 - \epsilon f_{n_i}(\zeta_0^{n_i})}{\beta_{n_i}'(0)} \quad \text{and} \quad s_{n_i}(t_{\zeta_1^{n_i}}) = \frac{C_{n_i}^2 - \epsilon f_{n_i}(\zeta_1^{n_i})}{\beta_{n_i}'(0)}$$

that tend to infinity when i goes to infinity. It follows that $s_{n_i}(\tau_{n_i}) \to \infty$; using $c_n s_n' - c_n' s_n = 1$, this yields $c_{n_i}'(\tau_{n_i}) \to 0$. Now, since κ_{n_i} has constant sign between θ_{n_i} and $\zeta_1^{n_i}$, $|c_{n_i}'(t)|$ reaches its maximal value at $t = \tau_{n_i}$; this gives for all $t \in [\theta_{n_i}, \zeta_1^{n_i}]$, $|c_{n_i}'(t)| \le |c_{n_i}'(\tau_{n_i})|$. From the mean value theorem, there exists $\mathfrak{a}_i \in [\theta_{n_i}, \zeta_1^{n_i}]$ such that $c_{n_i}(\theta_{n_i}) - c_{n_i}(\zeta_1^{n_i}) = c_{n_i}'(\mathfrak{a}_i)(\theta_{n_i} - \zeta_1^{n_i})$. Since $\theta_{n_i} - \zeta_1^{n_i}$ is uniformly bounded, we can make $c_{n_i}(\theta_{n_i}) - c_{n_i}(\zeta_1^{n_i})$ the smallest possible by setting i large enough. Now from $c_{n_i}(\theta_{n_i}) > c_{n_i}(0) = 1$, we see that provided i is large enough, $c_{n_i}(\zeta_1^{n_i}) > 0$, so c_{n_i} does not vanish on $[0, \zeta_1^{n_i}]$, contrary to our assumption. This gives $c_n(t_0^n) > 0$ for n large enough, and in the cases (1) and (3), we also get $c_n(t_1^n) > 0$, making $c_n(t_0^n) + c_n(t_1^n)$ positive near the boundary.

We are left with the cases (2) and (4). We already know that $c_n(t_{\zeta_1^n}) > 0$ near the boundary, and want to prove that $c_n(t_1^n) > 0$. We look at the set $A = \{c_n(t_1^n), c_n(t_1^n) < 0\}$. If A is finite, there is nothing to do; if it is infinite, denote its elements by $\{c_{n_i}(t_1^{n_i})\}$ and write $c_{n_i}(t_1^{n_i}) - c_{n_i}(\tau_{n_i}) = c_{n_i}'(\mathfrak{a}_i)(t_1^{n_i} - \tau_{n_i})$, where $\mathfrak{a}_i \in [\tau_{n_i}, t_1^{n_i}]$. We have $0 \ge c_{n_i}'(t_1^{n_i}) = \frac{c_{n_i}(t_1^{n_i}) s_{n_i}'(t_1^{n_i}) - 1}{s_{n_i}(t_1^{n_i})} > -\frac{1}{s_{n_i}(t_1^{n_i})}$. Like before, $s_{n_i}(t_1^{n_i})$ goes to infinity when $i \to \infty$, hence $c_{n_i}'(t_1^{n_i}) \to 0$. Now, since $|c'(t)| \le |c_{n_i}'(t_1^{n_i})|$, $\forall t \in [\tau_{n_i}, t_1^{n_i}]$ and $t_1^{n_i} - \tau_{n_i}$ is bounded, we can make $|c_{n_i}(t_1^{n_i})|$ the smallest possible. The proof is completed by observing that actually $c_n(t_0^n) > 1$ for n sufficiently large. To do this, consider the set $B = \{c_n(t_0^n), t_0^n > \theta_n\}$. If B is finite, the assertion follows, for c_n is an increasing function on $[0, \theta_n]$. In the same manner, if B is infinite, we exploit the fact that $c_n(t_0^n)$ is positive, and make it the closest possible to $c_n(\theta_{n_i})$. Since $c_n(\theta_{n_i}) > 1$, the conclusion follows. $\qquad\square$

Theorem 5.23 *Let (T, K) be a torus in the family given in Theorem 4.19. We exclude the case in which there are open sets of constant curvature. Assume f satisfies the following conditions:*

(1) κ has simple zeros;
(2) There is only one critical orbit of K in each band of the torus;
(3) For $\epsilon = \pm 1$, there exists x_0 in which f has sign ϵ, such that:

$$-\epsilon f''(x_0) \left(\int_{x_0}^{x_1} \frac{dx}{\sqrt{M_\epsilon - \epsilon f(x)}} \right)^2 < 2\pi^2,$$

and

$$\epsilon f''(x_{cr}^{-\epsilon}) \left(\int_{\zeta_0}^{\zeta_1} \frac{dx}{\sqrt{\epsilon f(x_0) - \epsilon f(x)}} \right)^2 < 2\pi^2,$$

where $x_{cr}^{-\epsilon}$, ζ_0, ζ_1, x_1 and M_ϵ are as in propositions 5.20 and 5.21.

Then, there is a neighborhood $\mathcal{N}$ of the torus in $\mathcal{L}_K(T)$ such that the metrics in $\mathcal{N} \cap \mathcal{SL}_K(T)$ have no conjugate points, whereas the others admit conjugate points.

Proof Denote by g_0 the metric on T; g_0 has no conjugate points. The assumptions on g_0 imply that a neighborhood of g_0 in $\mathcal{L}_K(T)$ can be taken in $\mathcal{L}_K^*(T)$. By propositions 5.20 and 5.21, conditions (1) and (3) imply that the extended function $\mathcal{Z}(g_0)$ is strictly positive on the open set

$$\widetilde{\Omega}(g_0) = \{p \in T, \ g_0(\nabla_{K_0} K_0, \nabla_{K_0} K_0)(p) \neq 0\}.$$

Therefore, there exists a neighborhood of any compact subset of $\widetilde{\Omega}(g_0)$ in

$$\widetilde{\Omega}(g) = \{(g, p) \in S\mathcal{L}_K^*(T) \times T, \ g(\nabla_K K, \nabla_K K)(p) \neq 0\}$$

on which $\mathcal{Z}$ is positive. Thus, we get a neighborhood V of g_0 in $\mathcal{L}_K^*(T)$ such that $\mathcal{Z}$ is positive on $SV \times F$, where $SV = V \cap S\mathcal{L}_K^*(T)$ and F is a compact subset of T not containing the critical orbits of K_0. We can make this compact subset the closest possible to the boundary. Now, the condition $f' f''' \leq 0$, combined with (1), implies that κ has exactly two simple zeros in the smallest period of f. Besides, these zeros cannot be on the extremums of f unless the torus is flat (see the proof of Lemma 5.19). Therefore, Lemma 5.22 provides a neighborhood V' of g_0 in $\mathcal{L}_K(T)$ such that $\mathcal{Z}$ is positive on $V' \times U$, U being a neighborhood of the critical orbits of K_0 in the torus. Taking $V \cap V'$ achieves the proof. $\qquad\square$

Theorem 5.23 gives a way to obtain examples of Lorentzian metrics on T with no conjugate points, that are stable by deformation in $S\mathcal{L}_K^*(T)$. Here are a few examples:

- The Clifton–Pohl torus, corresponding to $f(x) = sin(2x)$;
- $f(x) = \frac{sin(x)}{10+sin(x)}$;
- $f(x) = ln(2 + sinx)$;
- $f(x) = cos(sin(x)) - 3/4$;
- $f(x) = Jacobi\,SD(x, 1/2)$;
- $f(x) = Jacobi\,SN(x, 1/4)$.

The verification of hypothesis (3) of Theorem 5.23 is done numerically.

Remark 5.24 The quadratic variations of the Clifton–Pohl torus are also stable by deformation in $S\mathcal{L}_K^*(T)$; this comes from an explicit resolution of the Jacobi equation.

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